Spherical Harmony:
A Journey of Geometric Discovery
The harmonic relationship of circles, cones and spheres.
Full Edition

Gary Doskas

Editor

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Comments on "***Spherical Harmony***"

"The history of mathematics is replete with accounts of the discovery of new ideas in algebra or new aspects in geometry that no one before has ever investigated. The names of Pythagoras, Archimedes, Kepler, Galileo, Newton, and Cantor, to name only a few, are among those whose names live on with their discoveries in mathematics. Spherical Harmony, by Gary Doskas, belongs, in my estimation, in the category of new aspects in geometry.

It is also well known that new ideas in mathematics often find that they have applications in physics. It must be left for this new aspect in geometry, namely for polyconix, to find possible acceptance when applied in the realm of physics. "

- Father Magnus J. Wenninger OSB

Monk, mathematician, and builder of polyhedrons and polytopes

"I find the creative mathematical discoveries of Gary Doskas of the utmost originality and I predict that "Spherical Harmony" will add new value to the body of knowledge established by:

- RB Fuller of "Synergetic Geometry" fame – two books
- HSM Coxeter on "Regular Polytopes"... and
- Magnus J. Wenninger on "Spherical Models"

His findings on the relation of PI to the conical ratios found in spheres is of monumental value to the mathematics underpinning geometry."

William S. Becker

– UIC Professor of Industrial Design (ret.) and co-inventor

of the EarthStar Globe/Map geometry

Table of Contents

Preface

Several months ago, I became interested in the simple, yet strong structure of a tetrahedron. Believing that tetrahedrons would make good building blocks, I used modeling clay to create several equally sized tetrahedrons. I began building different shapes and before long the shape of an icosahedron surfaced, a shape I wasn't very familiar with. I couldn't help but notice how well the icosahedron would fit into a sphere and this sent me on a journey of geometric discovery.

As I started drawing circles on a sphere, I noticed that geometry behaved differently from what I learned in school. This intrigued me, and I was drawn into this non-Euclidean geometric world. I soon discovered some fascinating relationships between the circles, the sphere itself and all of the Platonic solids. I wasn't solving equations or calculating any measurements. I was just doing trial and error geometric construction using a compass, a lead pencil and a sphere.

I began to see these relationships more clearly, not with my eyes, but with my mind's eye. I felt no need to take out the calculator or to begin putting any of this in equation form, because the shapes themselves said everything. Being able to create these shapes by hand and hold them in front of me and experience them made learning fun and exhilarating. I didn't hesitate to act on my intuition as what to do next and I wasn't afraid to make a mistake. In fact I was learning just as much from my mistakes as I was from my breakthroughs. In retrospect, it was like finding the knowledge within. I was not inventing or creating anything, I was just uncovering that was always there, almost like recalling a memory.

I drew hundreds of patterns and developed my own tools to make construction easier. I wanted to learn more about this subject, so I searched the internet and visited libraries for some background material on this subject, but I couldn't find anything. So I began writing this book to capture what I was discovering. I use the word "harmonic" to describe these relationships, as this was the first word that came to mind as I was exposed to these beautiful shapes. To me they appear musical in nature. By musical, I'm not implying there is any sound involved, but the conix seemed to arrange themselves, based on simple, distinct ratios.

Non-Euclidean geometry is relatively young, discovered separately in the 1830s by mathematicians János Bolyai Nikolai and Ivanovich Lobachevsky, but it was never strongly embraced by the mainstream sciences. In fact, they both had difficulty getting the academic community to take their work seriously were only posthumously recognized for the incredible discoveries they had made.

Soon after the recognition of this discovery, spherical geometry was being used to more accurately circumnavigate the globe, which reduced travel time and fuel consumption of seafaring vessels. However, with the advent of World War II, the invention of computer technology and the hand calculator, spherical geometry was rendered to the sidelines again and interest waned.

I'm convinced that non-Euclidean geometry, more specifically, spherical geometry, is the purest form of geometry there is but unfortunately underutilized. After all, we live on a world that is primarily spherical. Our civilization owes so much to what Euclidean geometry has enabled for humanity for the last several thousand years. Spherical geometry is based on a rich, multi-dimensional world, which may imply complexity to some, but its greatest strengths are its simplicity and beauty and perhaps it will carry us forward, well into the future.

What is remarkable about these geometric relationships is that they can be experienced through pure geometric construction using just a compass. This can easily be experienced by both the young and mathematically inexperienced. Knowledge of spherical trigonometry is NOT required and only basic trigonometry is required to explore the more complex relationships in more detail.

This book is intended to be a beginning— a documentation of my own amateur "discoveries," put forth with innocent intentions to allow a wider audience and for those more experienced with this type of geometry to debunk or encourage my work. I want to continue to experiment, but only with the knowledge that I am not covering ground that has already been sufficiently explored. For that, I need input; indeed, I genuinely welcome it.

Prerequisite:

Before reading this book, I recommend the reader become familiar with the main features of the five Platonic solids. They are named for Plato, the ancient Greek philosopher and mathematician and have been known for thousands of years. Platonic solids are a convex polyhedron with regular polygon faces where the same number of faces meets at each vertex.

Every polyhedron has a dual polyhedron where the faces and vertices are interchanged. The dual of a Platonic solid is another Platonic solid, as follows:

The tetrahedron (it is a self-dual).

The octahedron and its dual is the hexahedron (or cube).

The hexahedron (or cube) and its dual is the octahedron.

The icosahedron and its dual is the dodecahedron.

The dodecahedron and its dual is the icosahedron.

The five Platonic solids are illustrated in the following Figures I-V.

The tetrahedron is a three-dimensional geometric solid composed of four equilateral triangular faces, four vertices and six edges.

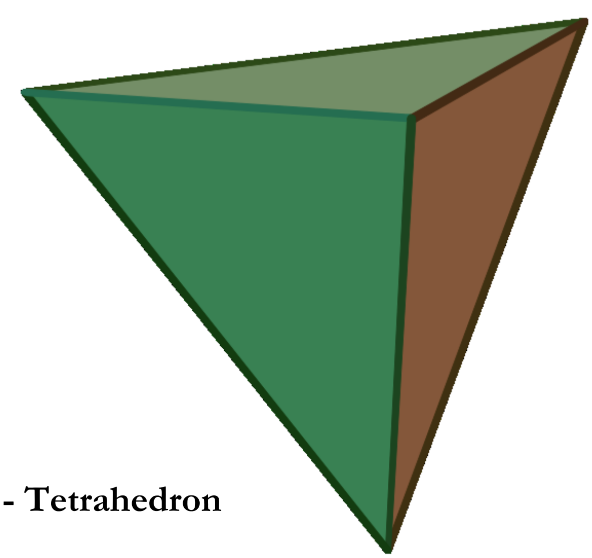


Figure I - Tetrahedron

The hexahedron or cube is a three-dimensional geometric solid composed of six square faces, eight vertices and twelve edges.

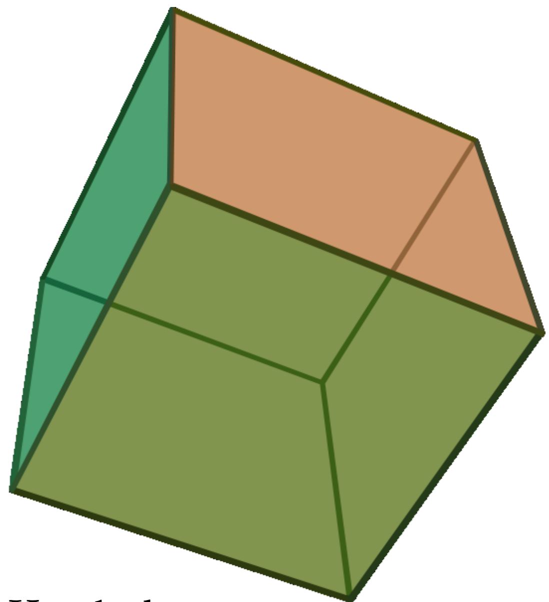


Figure II - Hexahedron

The octahedron is a three-dimensional geometric solid composed of eight equilateral triangular faces, six vertices and twelve edges.

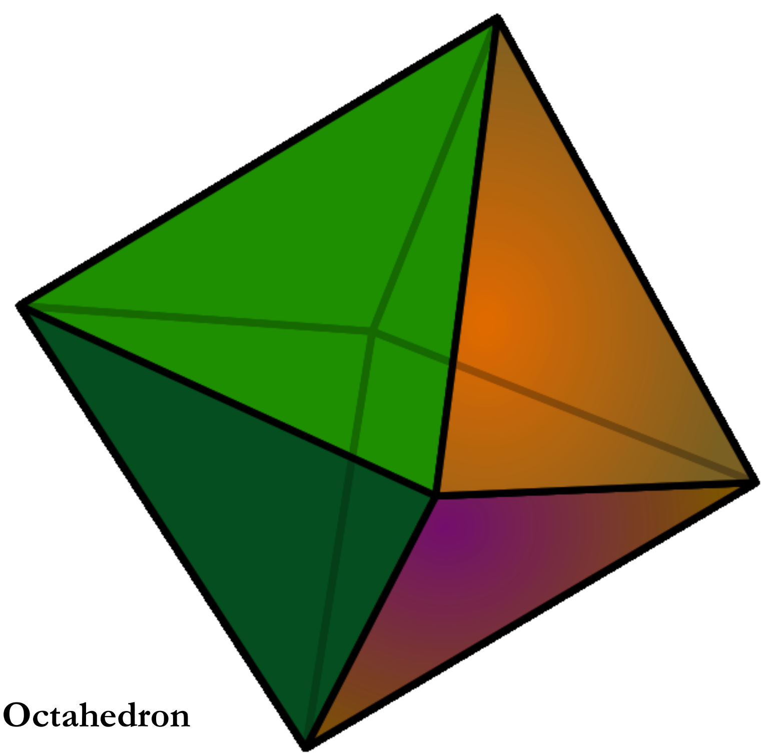


Figure III - Octahedron

The dodecahedron is a three-dimensional geometric solid composed of twelve pentagon faces, twenty vertices and thirty edges.

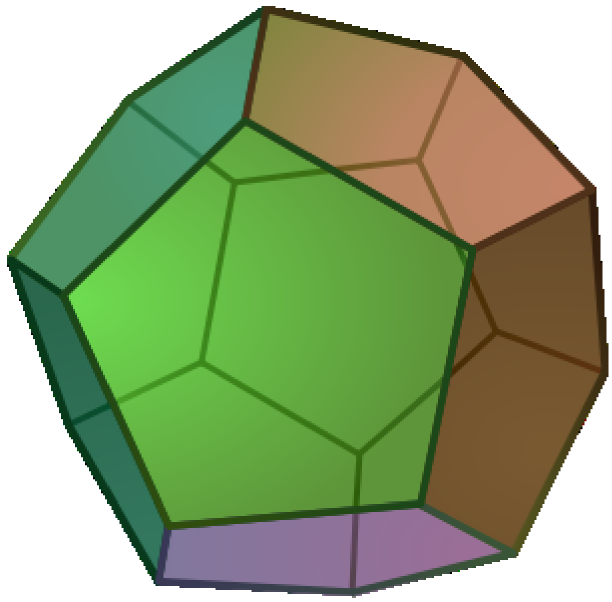


Figure IV - Dodecahedron

The icosahedron is a three dimensional geometric solid composed of twenty equilateral triangular faces, twelve vertices and thirty edges.

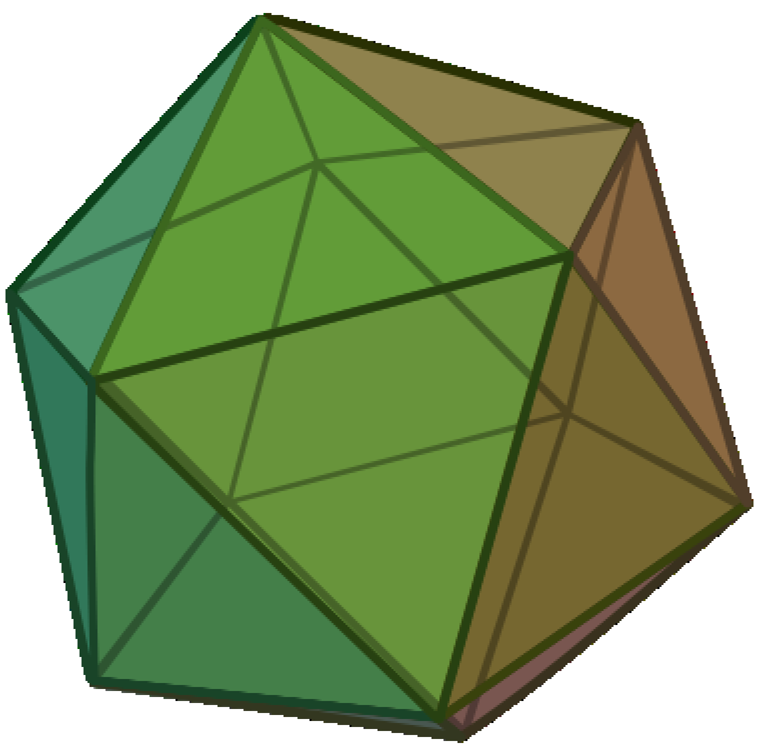


Figure V - Icosahedron

The next prerequisite is having a good understanding of the geometric relationships between a circle and its radius as defined by Euclidean geometry. There is the obvious $C = 2\pi R$ relationship. From a geometric construction standpoint, the radius R can be used to mark off six chords of length R which divides the circle's circumference into six equal arc segments. These six chords also define a hexagon inscribed in the circle. These relationships hold true for circles of all sizes. Refer to Figure A for a description of how this is done.

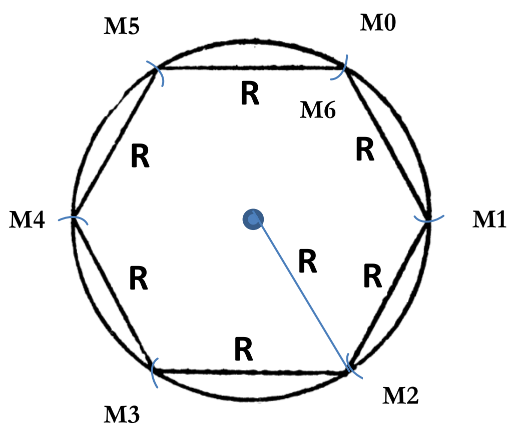


Figure A

To start, set the compass to some value R and draw a circle. Now make an arbitrary mark M0 on the circle. Place the compass at M0 and make a second mark at M1. Move the compass to M1 and make a third mark at M2. Continue in this fashion and the sixth mark M6 will coincide with the first mark M0 and as you can see in Figure A the circle has been divided into six equal arc segments. The six chords defined by these markings, create an inscribe hexagon. This is the natural relationship between a hexagon and a circle and is the basis for the "Flower of Life" (a symbol of sacred geometry which also has many spiritual and religious beliefs associated with it).

Refer to Figure A as the starting point to draw the Flower of Life. Draw six circles centered on points M0 through M5. The result can be seen in Figure B where these six new circles created a six-petal flower design that is also known as the "Seed of Life."

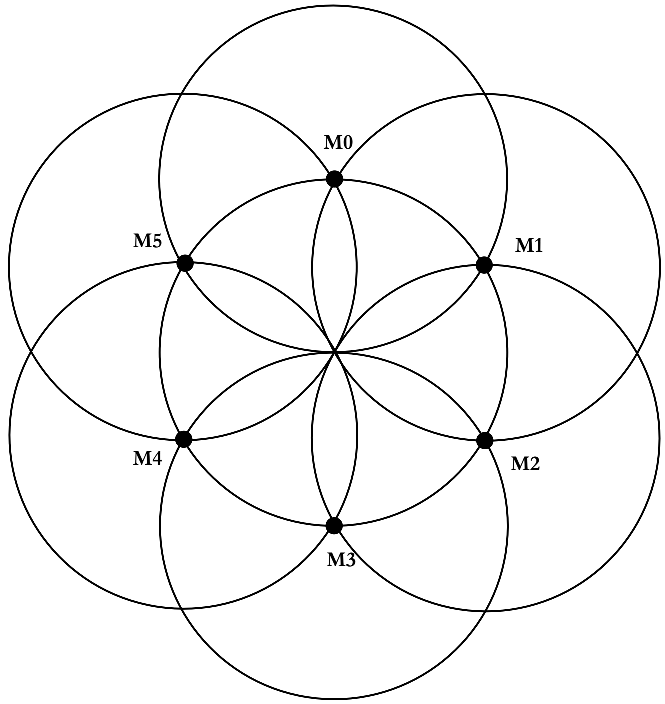


Figure B

Additional circles can be added to this arrangement by centering each new circle at the new intersection points between the outer circles which creates the Flower of Life as seen in Figure C.

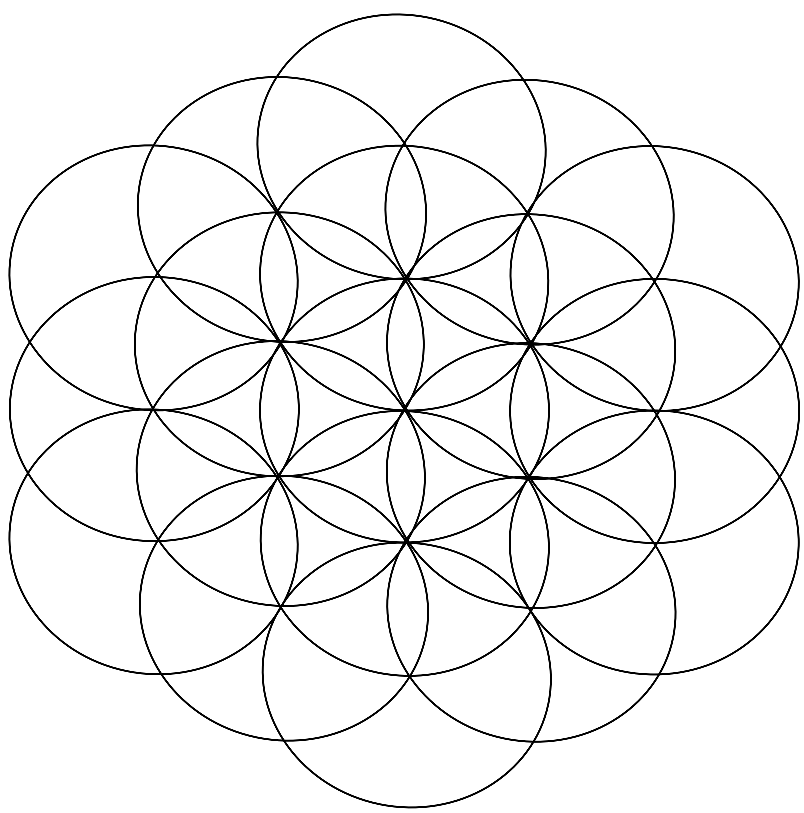


Figure C

Circles can continue to be added and this pattern would continue growing indefinitely. (Refer to Figure D. This is a form of geometric tiling where the circles naturally self-intersect and identify where to center additional circles in a self-guiding fashion.

When I first started drawing circles on a sphere, I thought I would be able to draw this Flower of Life pattern, however I was in for a big surprise, as you will see in the following chapters.

In Chapter 1, I describe the discovery of some fascinating relationships between circles, a sphere itself and all of the Platonic solids.

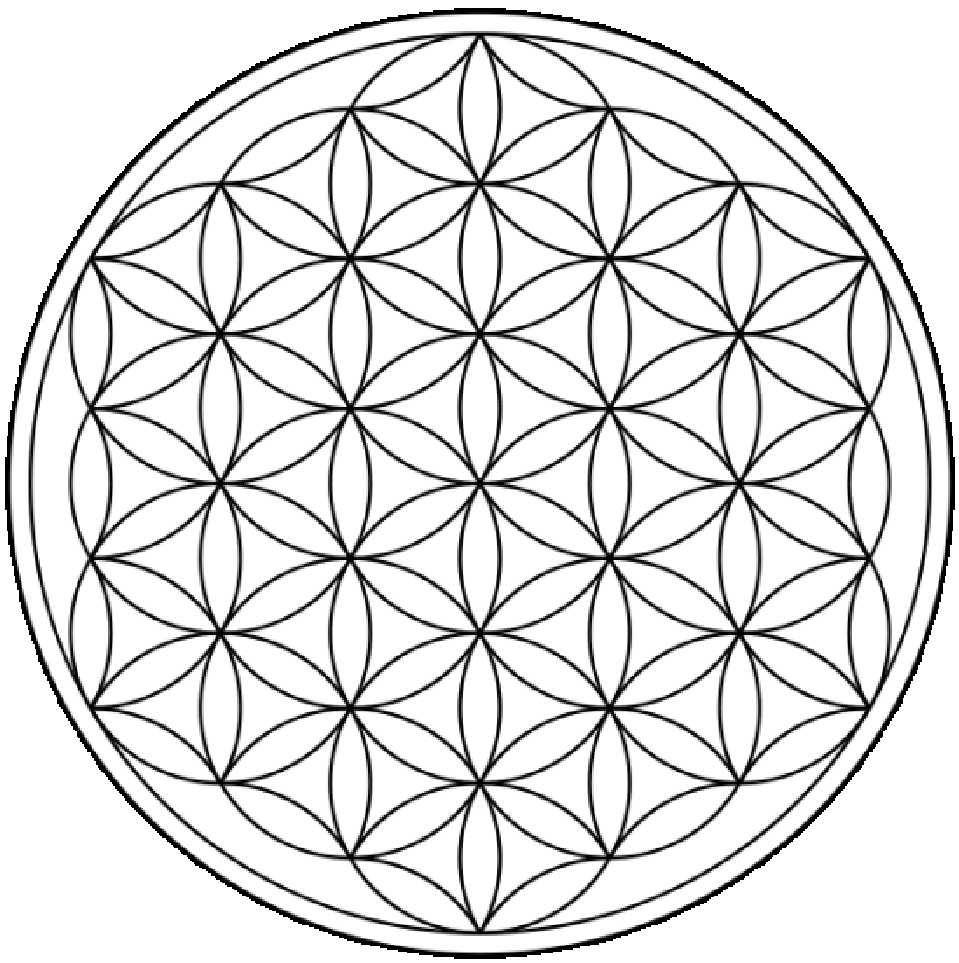


Figure D

Chapter 1: Introduction to Conix Inscribed on a Sphere

The subject matter in this book is not based on Euclidean geometry, but is based on *spherical* geometry (a form of non-Euclidean geometry). The primary focus of this book is to describe some simple yet fundamental geometric relationships between circles and cones, and the sphere they are inscribed within. As far as I know, these relationships have remained unknown for thousands of years. What I find fascinating is that these geometric relationships can be explored and revealed using just a compass and sphere.

The first part of the book is based on the geometric construction techniques used to discover these unique geometric relationships. This is followed by a more formal geometric proof which surprisingly, is not overly complex. For the rest of this book, I will refer to circles drawn on a sphere as a "**conix**" to differentiate it from a circle drawn on a plane.

(Note: I use the term "**conix**" as both the singular and plural form of a circle on a sphere.)

Before we proceed to construct these conix, a quick review of some of the basic properties of a conix could be helpful. As seen in Figure 1, the sphere has its center at point S with a radius of R. In this example the conix is constructed by placing the compass point on the sphere's north pole or its top antipodal center C_{ta} with a conical radius R_{ta}. A similar conix could also be constructed by placing the compass point on the sphere's south pole or its bottom antipodal center C_{ba} with a conical radius R_{ba}. The planar center C_p of the conix resides on the axis of the sphere and has a planar radius of R_p. The poles and center of the sphere are all, of course, lined up on the sphere's central axis. Euclidean geometry applies to the relationship between the conix and its planar radius R_p. However, Euclidean geometry does not apply to the relationships between the conix and its conical radii R_{ta} and R_{ba}.

It's this unique relationship between the conix and its conical radii and the sphere that this book will be focusing on. I should also point out that the conix in Figure 1 can also be described as the intersection of a cone and the sphere. There are three cones (N_{ta}, N_{ba} and N_s) that share the conix at its base circle, which have their apex located at points C_{ta}, C_{ba} and S respectively. The aperture of cone N_{ta} is 2Ω. The aperture of cone N_{ba} is 2Φ. The aperture of cone N_s is 2Θ. The aperture of a right circular cone is the maximum angle between two lines on the cones surface.

The cone acts like a bridge that links the geometric properties of the two dimensional circle with the three dimensional sphere. A right angle cone, in essence defines one of two spheres. In one case, the apex of the cone is located on the sphere's surface (C_{ta} or C_{ba}) and in the other the apex is located at the center (S) of the sphere. This relationship between the cone and the sphere will be explored in the later chapters.

Here are some of the basic properties of the conix for the geometric model that will be used in this book:

i) There is an infinite number of conix on the surface of a sphere where the planar centers C_p lay on the axis of the sphere, and where R_p is the planar radius. Euclidean geometry traditionally applies to the relationship between C_p and R_p.

ii) Every conix has two conical centers (C_{ta} and C_{ba}) at the antipodal points on the axis of the sphere, with conical radii R_{ta} and R_{ba}. Euclidean geometry traditionally does *not* apply to the relationship between C_p and R_{ta} and R_{ba}.

iii) Every conix in the northern hemisphere has a dual in the southern hemisphere.

iv) One bounding condition exists at the north pole of the sphere, where the conical angle Ω approaches $\pi/2$ and the conical radius R_{ta} approaches zero.

v) One bounding condition exists at the south pole of the sphere, where the conical angle Ω approaches zero and conical radius R_{ta} approaches two.

$R_{ba} = 2R\cos\Phi$

$R_p/R_{ba} = \sin\Phi$

$R_{ta} = 2R\cos\Omega$

$R_p/R_{ta} = \sin\Omega$

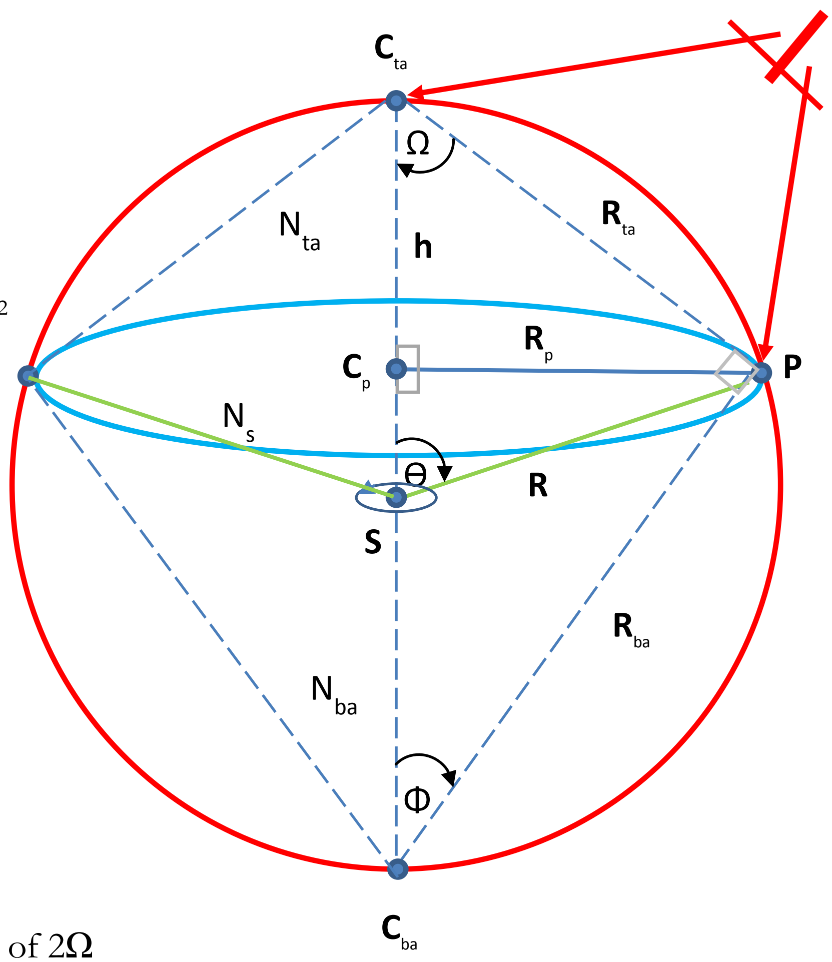
$4R^2 = (R_{ta})^2 + (R_{ba})^2$

$h = (R_{ta})^2/2$

$\Omega + \Phi = \pi/2$

$\Theta + 2\Omega = \pi$

$\Theta = 2\Phi$



Ranges:

$0 < \Omega < \pi/2$

$0 < \Phi < \pi/2$

$0 < \Theta < \pi$

$0 < R_{ta} < 2R$

$0 < R_{ba} < 2R$

$0 < R_p < R$

N_{ta} has an aperture of 2Ω

N_{ba} has an aperture of 2Φ

N_s has an aperture of 2Θ

Figure 1

It is important to understand the difference between circles drawn on a plane compared to a conix drawn on a sphere.

A circle drawn on a plane is a planar circle because its center, radius and circumference are all contained in the same plane and as a result it is a two dimensional object.

The conical radius used to draw a conix has or implies three dimensions, similar to that of its associated cone. It has the same two dimensions as a planar circle, plus it has a height component (h), which is the height of the cone. When you place the compass point on a sphere's north pole and start drawing a conix, you will notice that the sphere's surface drops away from the north pole. As a result, this adds the height component to the conical radius (R_{ta}).

How does the curvature of the sphere's surface affect the radii of a conix? For very small conix drawn on a very large sphere, R_p and R_{ta} are close to the same size, with R_p being slightly smaller. However, as the curvature of the sphere's surface increases, the planar radius R_p decreases proportional to the conical radius R_{ta}. The relationship between the conical radius R_{ta} and the planar radius R_p is illustrated in Figures 2-4, where the radius R of the sphere is kept constant.

For small radii, as seen in Figure 2, R_{ta} and R_p are similar in size and the height component **h** is relatively small. However, based on the Pythagorean Theorem, the conical radius R_{ta} can be proven to be slightly larger than the planar radius R_p. To put this in perspective, if a conix, one mile in diameter, is drawn on a sphere the same size as the earth, the height component **h** would be about two inches. The difference between R_{ta} and R_p would be only be about eight billionths of an inch in length

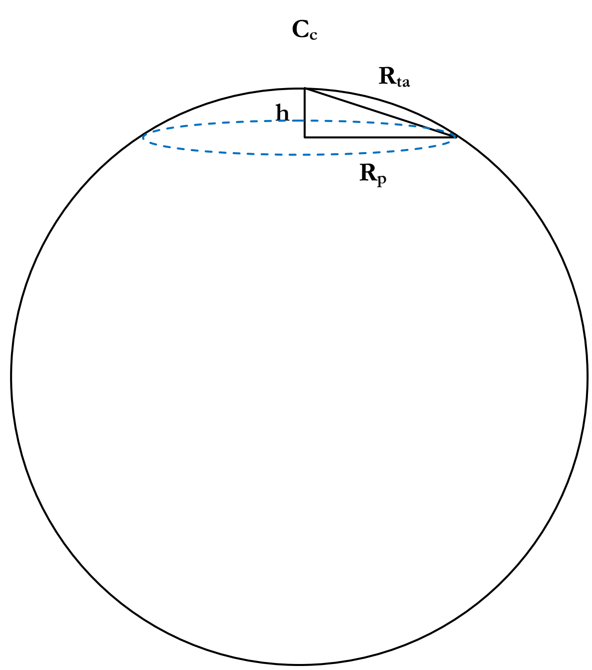


Figure 2

Observe in Figure 3, that as the conical radius R_{ta} increases, the planar radius R_p has comparatively decreased. You will also notice an increase in the height component **h**.

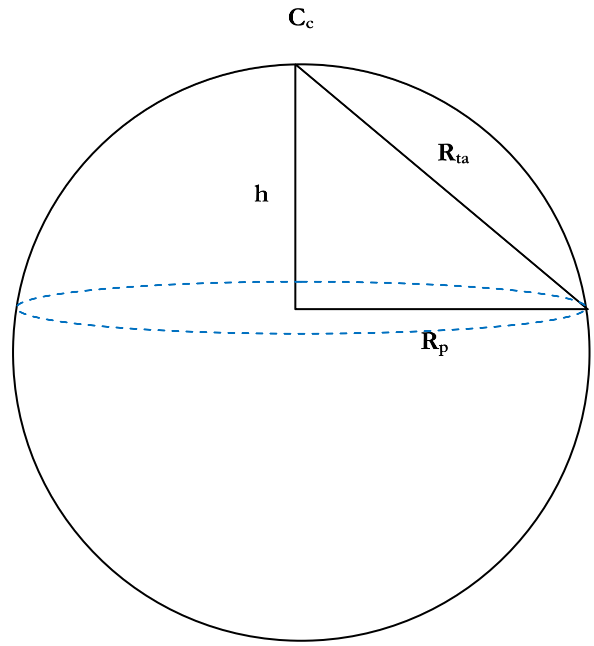


Figure 3

The conical radius R_{ta} has been increased further as seen in Figure 4. Now, R_{ta} reaches well into the southern hemisphere and the size of R_p has been dramatically reduced. In fact, as the conical radius approaches the length of the spherical diameter, the length of R_p would approach zero. If the conical radius R_{ta} exceeds the diameter of the sphere, a conix cannot be drawn.

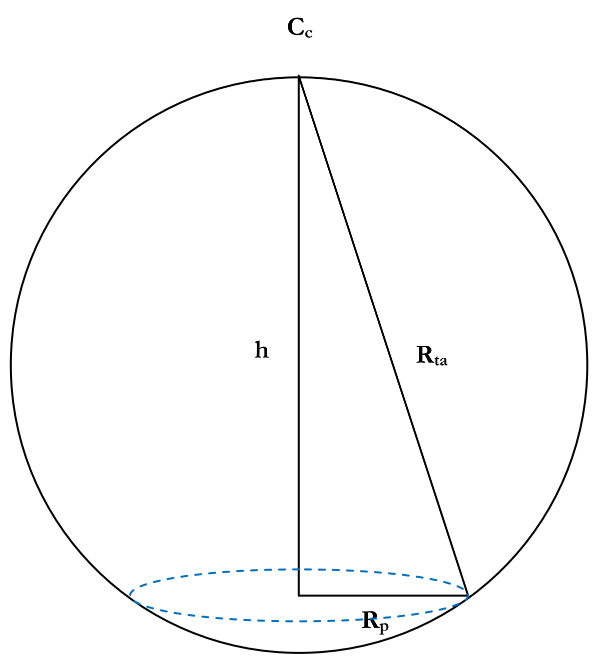


Figure 4

The length of the planar radius R_p and the size of the conix are a function of both the sphere's radius R and the length of the conical radius R_{ta}. The conix has some very interesting geometric properties because it's based on four different radii (R, R_{ta}, R_{ba} and R_p).

When constructing a planar circle, there is no physical limit to the size of the circle, other than being limited by the size of your compass. In spherical geometry, the size of the conix is limited by the size of the sphere. For analytic purposes, a unit sphere is all that is required. As a result, the conical radius only has a range from zero to two on a unit sphere.

It is important to note that the classical V-shaped compass is not sufficient to draw conix on a sphere, because it is difficult if not impossible to place the compass point on the north pole and draw a conix in the southern hemisphere. A spherical compass is required to draw a conix in any hemisphere on the sphere. Figure 5-6 illustrates this point, where the spherical compass in colored green, the conix is colored blue and the classical compass is colored red. For conix in the northern hemisphere, a classical compass can sometimes be used as seen in Figure 5, but as the conix approaches the equator, the compass points are not very perpendicular to the surface making it difficult to draw with.

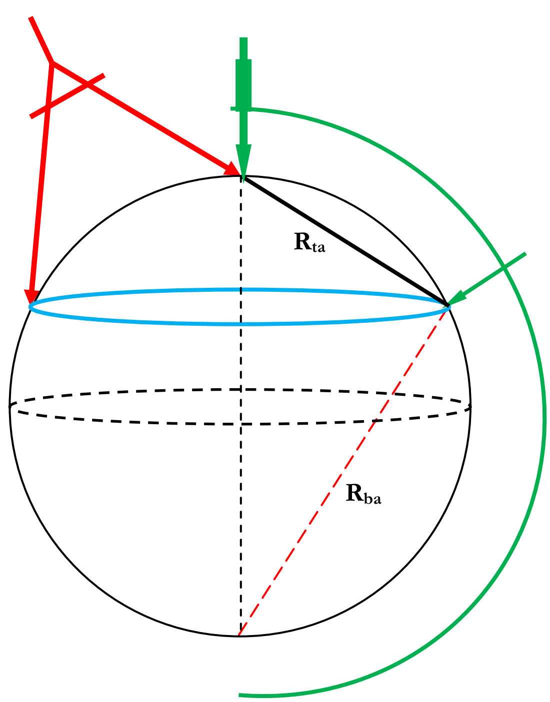


Figure 5

For conix drawn in the southern hemisphere, the problem only gets worse and a spherical compass is required; refer to Figure 6a.

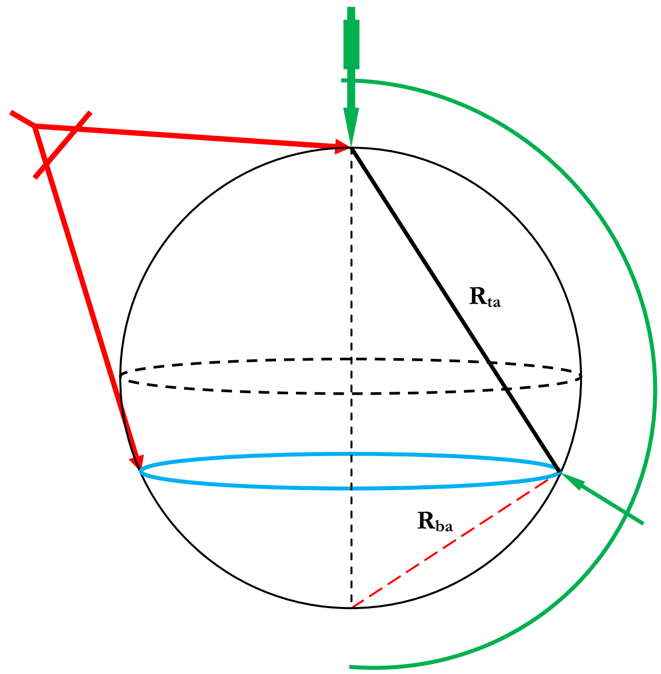


Figure 6a

Now being familiar with the basic properties of a conix, we can proceed to Chapter 2 and start constructing conix on the surface of a sphere.

Chapter 2: Constructing Conix on a Sphere

The first thing I discovered while drawing conix on a sphere, was that many of the relationships that are defined by Euclidean geometry do not apply. One of the most basic relationships in Euclidean geometry is that the radius of a circle can be used to divide its circumference into six equal arc segments and thus define an inscribe hexagon. This geometric relationship is the basis of the Flower of Life as described earlier in the Prerequisite chapter.

Before I proceed, I should describe what led me to this point in my journey. It started when I first held an icosahedron in my hands for the first time. I instantly envisioned the sphere that circumscribed it and felt compelled to geometrically locate its vertices. The following describes the steps I took to do this.

I started by drawing the initial conix centered at the north pole, refer to Figure 6b to see how an arc segment is marked on the first conix with a chord of length R_{ta}. Initially I assumed that I could use the conical radius R_{ta} to divide a conix into six equal arc segments as was seen in the Flower of Life. To my surprise, the conix was not divided into equal arc segments. This was both puzzling and disappointing at first and I assumed I had done something wrong, so I repeated the process only to get the same result. I soon realized that the conical radius R_{ta} that I chose to draw the conix, was too long to divide the conix into six equal arc segments because the sixth marking overshoots the start point. At first I tried reducing the size of R_{ta}, and although the overshoot was slightly decreasing, the conix was getting so small, it was difficult to draw.

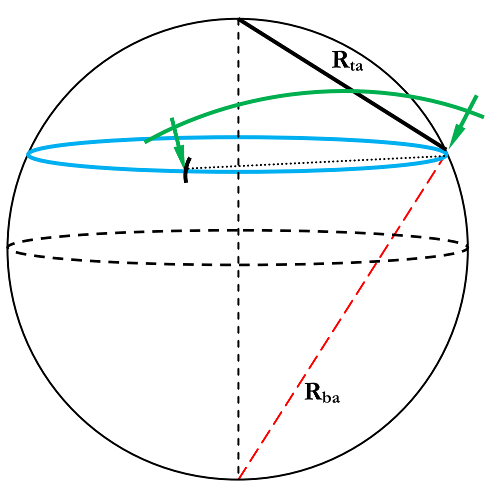


Figure 6b

Let me describe in more detail how I proceeded, refer to Figure 7 which is a North Pole view of a sphere (in red) and conix (colored blue). I placed the compass at point C_{ta} (or North Pole) and drew a conix with a conical radius of R_{ta}. I then made an arbitrary mark at M0 on the conix. I moved the compass to M0 and made another mark at M1 defining a chord of length R_{ta}. I then moved the compass to M1 and made another mark at M2. I continued marking off chords of length R_{ta}, until I returned to the starting mark M0. As you can see in Figure 7, M6 overshoots the M0 marking indicating that R_{ta} was too long to divide the conix into six arc segments.

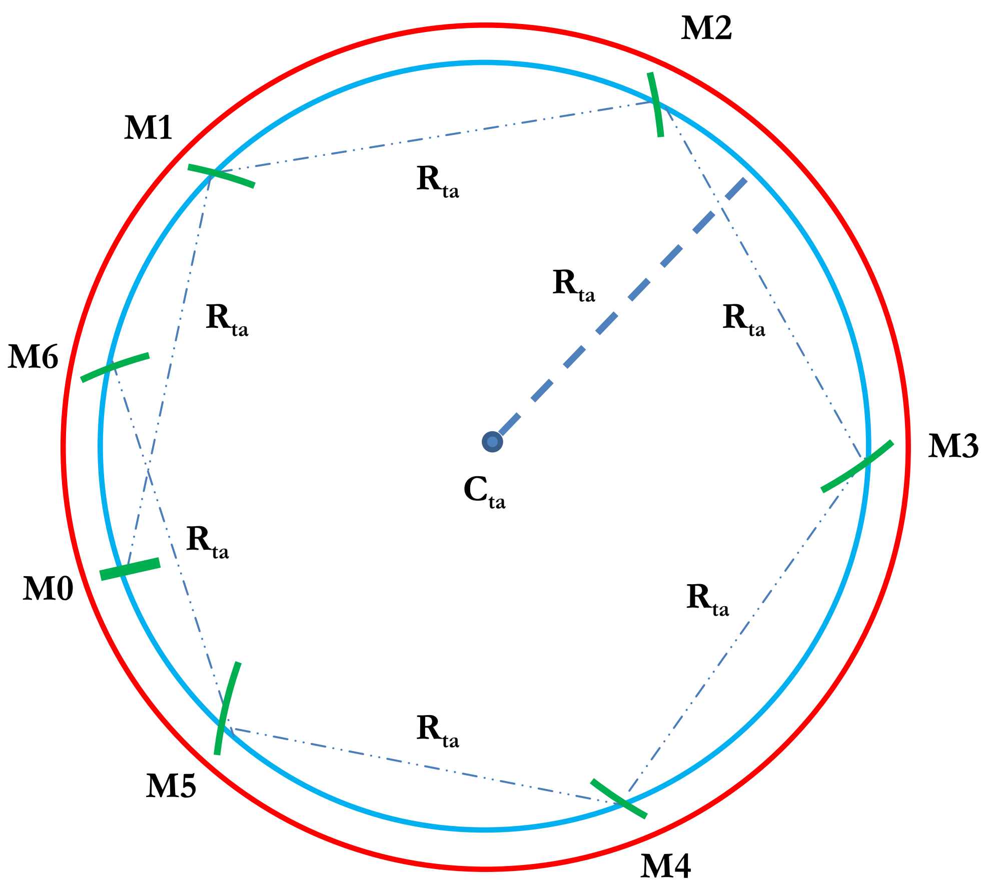


Figure 7

At this point, I increased the length of R_{ta} and redrew the conix and made similar markings on the conix. You can see in Figure 8 that the fifth marking M5 now undershot M0. I found out the hard way that any time you make an adjustment to R_{ta} you need to erase the old conix and start over. It is important to use the same radius for drawing the first conix as well as for marking off arc segments on the conix. It may take several iterations of tuning R_{ta} to the correct length and you may alternate between overshooting and undershooting the initial mark M0.

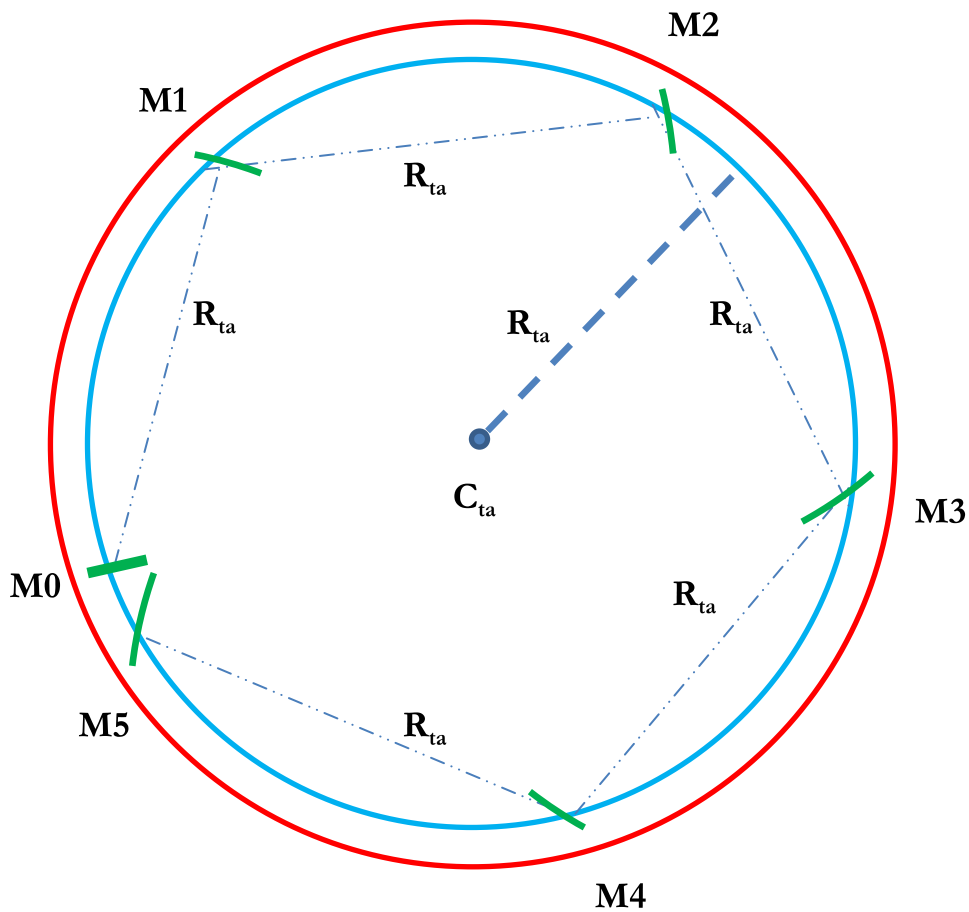


Figure 8

I continued to tune the conical radius R_{ta} and eventually the 5^{th} marking M5 lined up with the start point M0, and the conix was divided into five equal arc segments by an inscribed pentagon, as seen in Figure 9. This was my first EUREKA! moment. I didn't know where this was heading but it piqued my interest.

I came to the following conclusion:

"***There is only one conix on a sphere where its conical radius R_{ta} is the same size as the side of its inscribed pentagon"***.

I will refer to this conix as the 5^{th} harmonic conix.

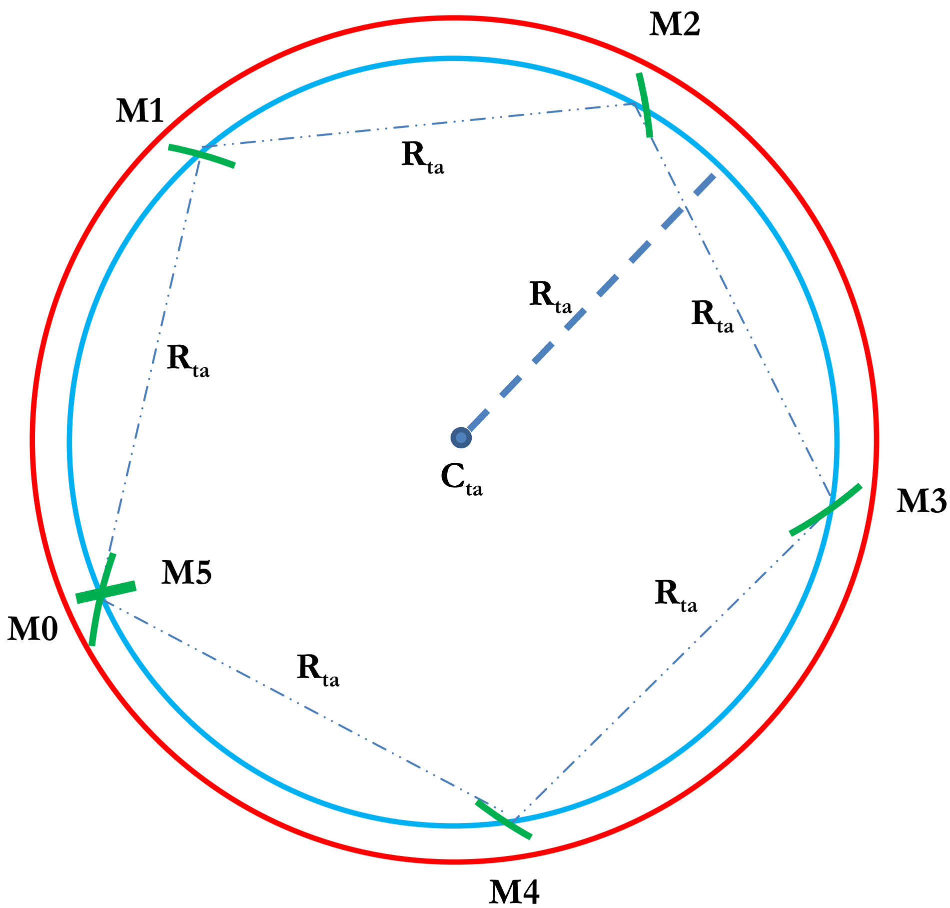


Figure 9

I felt compelled to continue drawing more conix at the intersection points created by each new conix constructed. I did this in a similar fashion as one would draw the "Flower of Life" as described in the Prerequisite chapter. Initially, I expected that there would an endless number of conix to draw. To my amazement after drawing twelve conix, the pattern closed in on itself, and became a seamless and overlapping pattern, oriented around multiple axes of the sphere. The pattern was similar to the geometric tiling we saw in the Flower of Life, except that it was based on a five-petal pattern instead of the six-petal pattern. What is remarkable is that the natural intersection of these harmonically coupled conix defined the vertices of an icosahedron, a classical Platonic Solid! It was EUREK AGAIN! I will refer to this arrangement of twelve conix as the 5th harmonic note, refer to Figure 10.

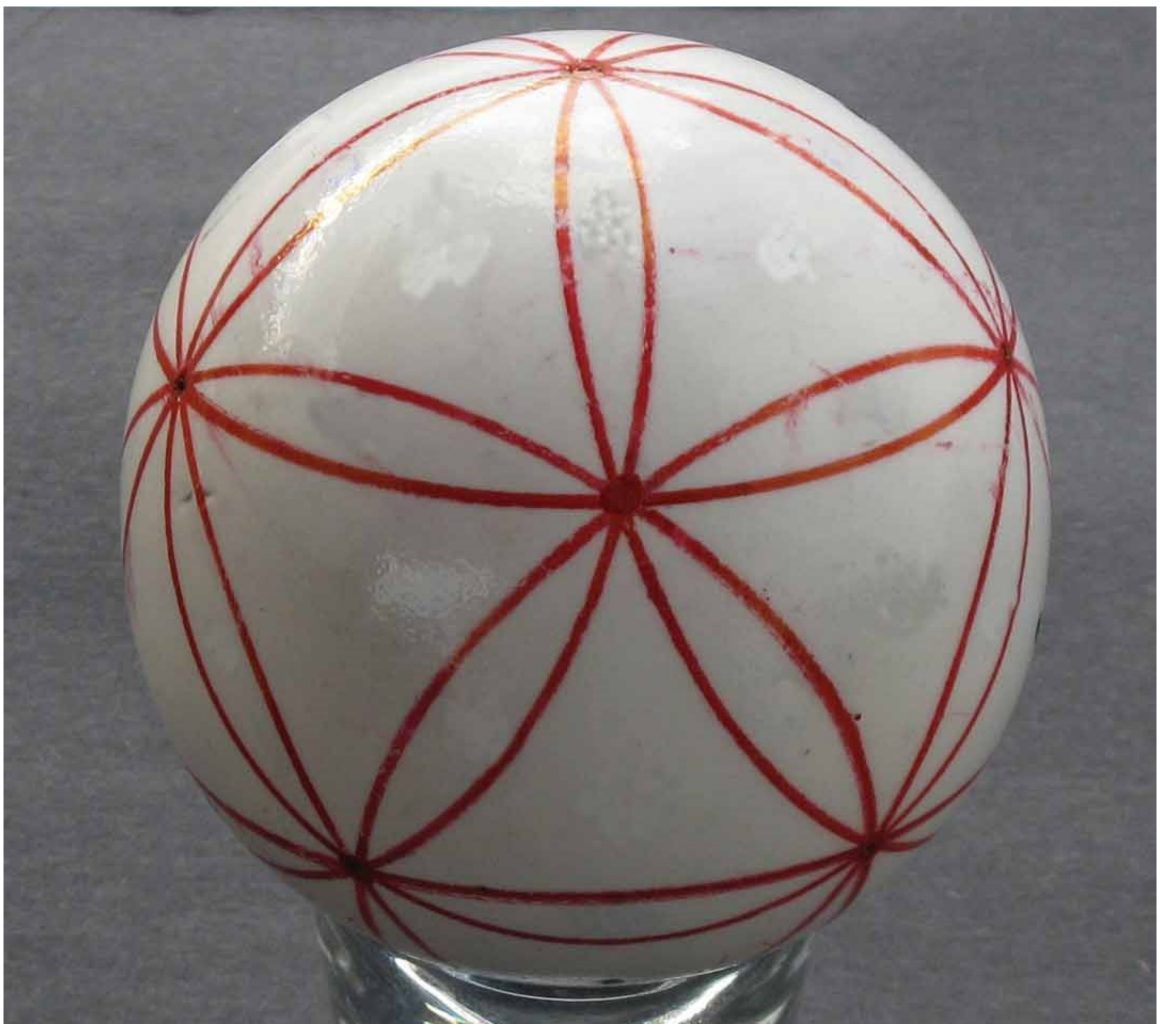


Figure 10

Upon discovering the 5th harmonic note and classical icosahedron, I spent weeks drawing them on any spherical surface I could find, including golf balls, basketballs and beach balls. It then dawned on me that there may be more harmonic relationships on a sphere. It seemed a natural progression that a 4th harmonic could be discovered. So using a similar approach as I used for the 5th harmonic (pentagon-based), I started increasing the length of R_{ta} to see where that would lead. It wasn't long before I discovered the 4th harmonic conix (square-based). As seen in Figure 11, R_{ta} was used to mark off four chords and define an inscribed square in the conix. The conix was also a great circle and divides the sphere into equal hemispheres. (See Figure 12.)

I came to the following conclusion:

"***There is only one conix on a sphere where its conical radius R_{ta} is the same size as the side of its inscribed square"***.

I will refer to this conix as the 4th harmonic conix.

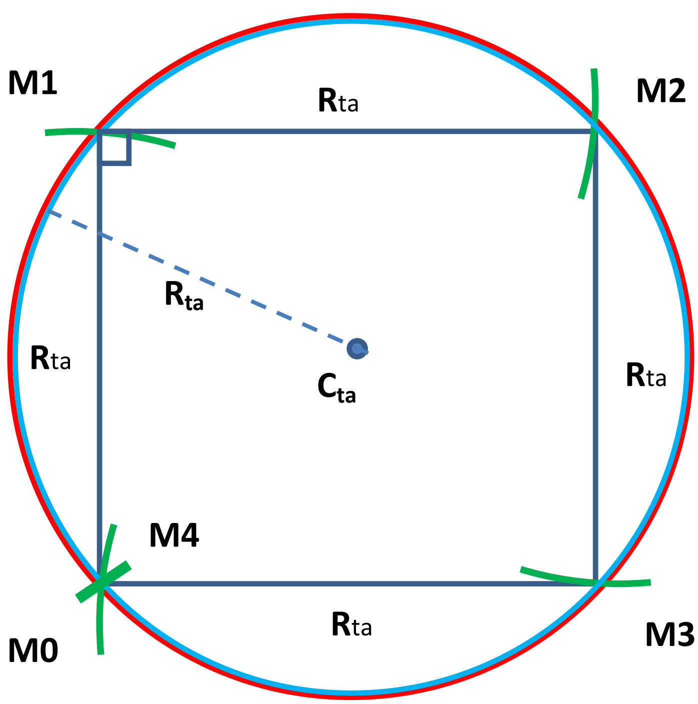


Figure 11

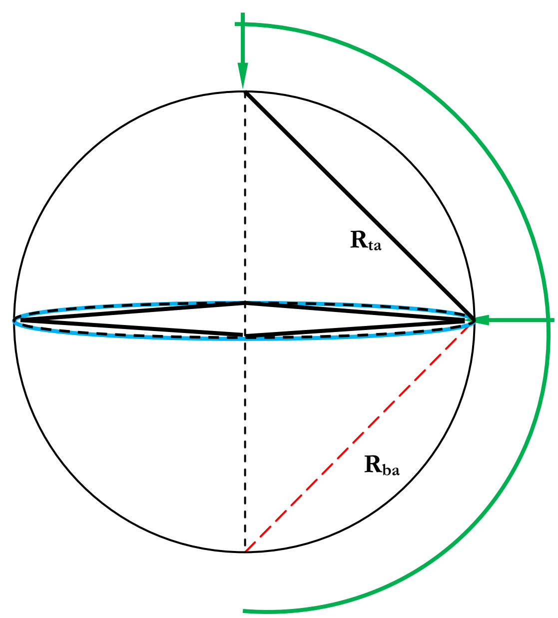


Figure 12

As I had done with the 5th harmonic conix, I took the Flower of Life approach and started drawing more conix centered on each intersection point. After drawing six conix, the natural intersection of these harmonically coupled conix defined the vertices of an octahedron, which will be referred to as the 4th harmonic note. (See Figure 13.) Although there appears to be only three conix in the 4th harmonic note, it actually consists of three pairs of overlapping conix. What appears as one conix, is actually a pair of conix centered on antipodal points on the sphere.

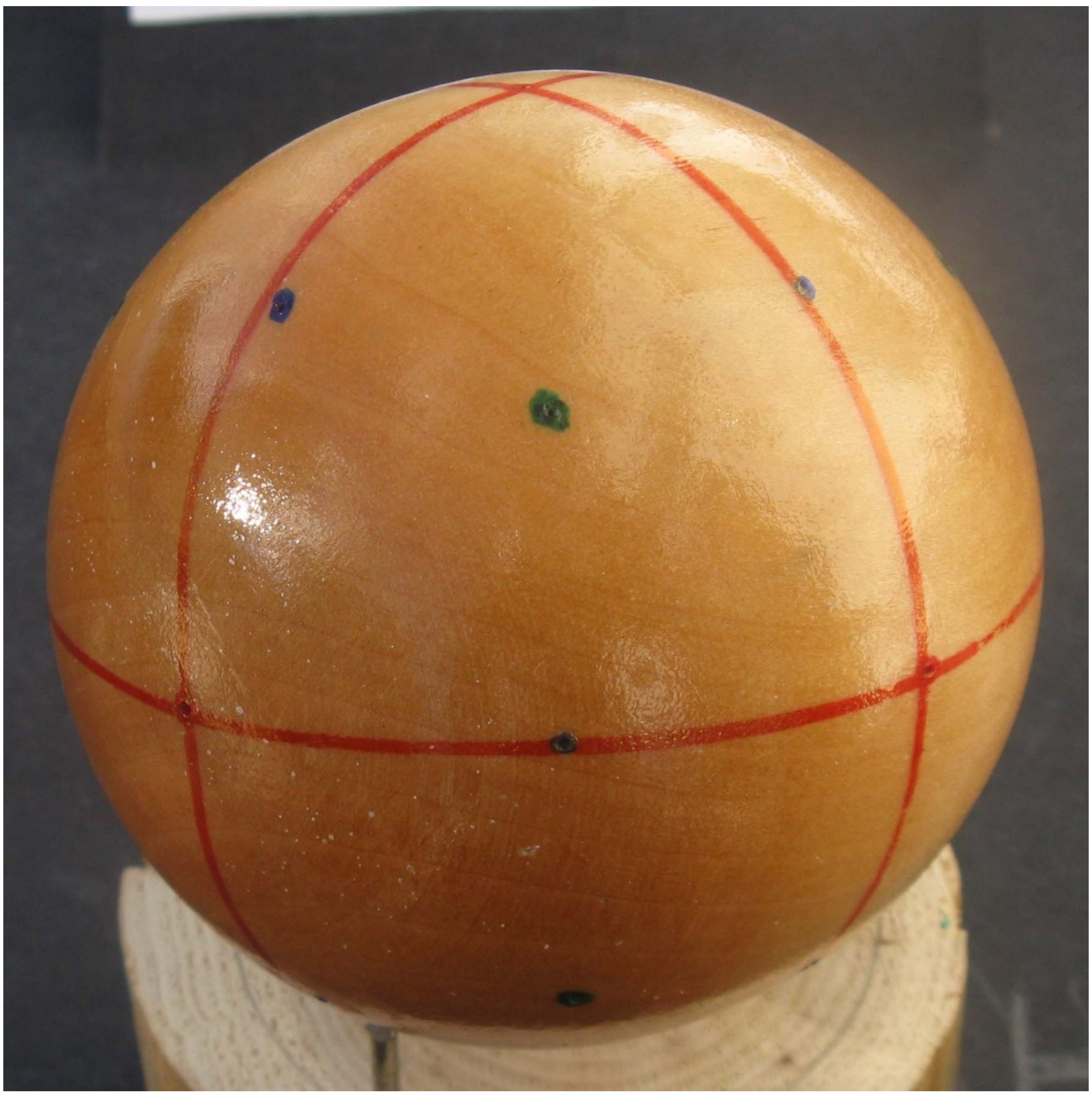


Figure 13

At this point I decided to attempt to derive a general solution that describes the various harmonic conix and its relationship to the sphere. I use the 5th harmonic example to develop the general solution. I knew that every conix on the sphere is divided into five equal arc segments by its inscribed pentagon, and the size of the conix is a function of its conical radius R_{ta} and its conical angle **Ω**. I needed to find an inscribed pentagon whose side was also equal to its conical radius R_{ta}. If I could find this pentagon, it would confirm what I found by construction. If I couldn't find this pentagon, it would have invalidated my construction proof. Refer to Figure 14 for the various geometric relationships that will be used in the general solution.

General Solution:

To find the values of **Ω** and $\mathbf{R_{ta}}$ for each harmonic note, I will define three equations that describe $\mathbf{R_{ta}}$ in terms of **Ω,** $\mathbf{R_p}$ and **w**, where **w** is the "chord angle" (2π/n) of the inscribed n-sided polygon.

Based on triangle $\mathbf{C_{ta}\,C_{ba}}$ **P4 (A right triangle based on Thales' Theorem)**

$$(E1)\ R_{ta} = 2R\cos\Omega$$

Based on triangle $\mathbf{C_{ta}\,C_p}$ **P4**

$$(E2)\ R_p = R_{ta}\sin\Omega$$

Based on the formula for a chord:

(Triangle $\mathbf{P1P2C_p}$)

$$(E3)\ R_{ta} = 2R_p\sin(w/2)$$

By substituting $\mathbf{R_p}$ from (**E2**)

$$(E4)\ R_{ta} = 2(R_{ta}\sin\Omega)\sin(w/2)$$

$$1/2\sin\Omega = \sin(w/2)$$

$$\sin\Omega = 1/2\sin(w/2)$$

$$(E5)\ \Omega = \sin^{-1}(1/2\sin(w/2))$$

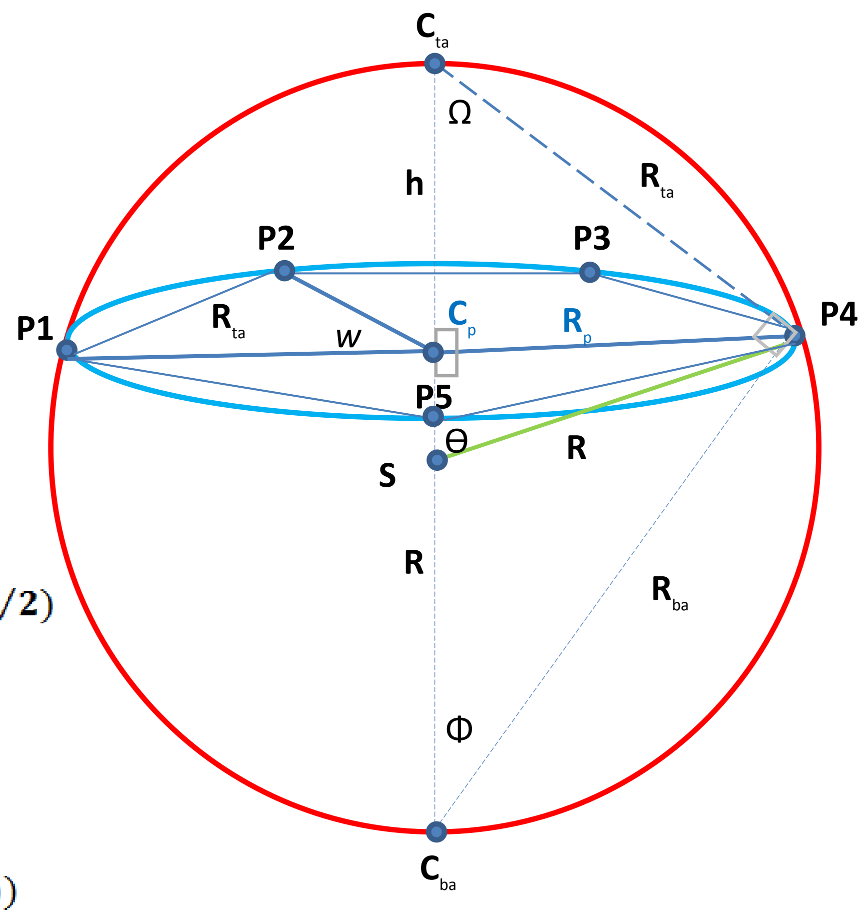


Figure 14

Now I can solve for the various harmonics because I know the chord angle **w** for the inscribed polygon in each conix and I can calculate the value of the conical angle **Ω** from equation (**E5**). Subsequently, the conical radius $\mathbf{R}_{ta}$ and planar radius $\mathbf{R}_{p}$ can be calculated from equations (**E1**) and (**E2**). The 5th harmonic's inscribed pentagon has **w** equal to $2\pi/5$. The 4th harmonic's inscribed square has **w** equal to $2\pi/4$.

(w = 2π/5) 5th harmonic $\quad$ **Ω ~= π/3.0884** $\quad$ **and** $\mathbf{R_{ta} = 4/\sqrt{(10+2\sqrt{5})}}$

(w = 2π/4) 4th harmonic $\quad$ **Ω = π/4** $\quad$ **and** $\mathbf{R_{ta} = \sqrt{2}}$

I noticed a trend where the number of sides of the polygon inscribed in each conix decreased by one as the harmonic decreased.

The 5th harmonic conix is based on an inscribed pentagon.

The 4th harmonic conix is based on an inscribed square.

I suspected that there could be a 6th harmonic conix with an inscribed hexagon. I knew that the chord angle **w** of an inscribed hexagon would be equal to **2π/6.** This would correspond to the 6th harmonic conix, if it existed. So I decided to test this value in the general solution, and it predicted theses values:

$\mathbf{\Omega = \pi/2} \qquad \mathbf{R_{ta} = 0} \qquad \mathbf{R_{ba} = 2}$

Interestingly, the 6th harmonic describes the infinitively small point at the north pole where no curvature exists, where the rules of Euclidean geometry apply. I'm not really sure what is implied by the 6th harmonic, but the general solution is hinting that it's of a very small dimension. In this geometric model, Euclidean geometry would be a good approximation for geometry drawn on a small patch earth of our planet.

It seemed reasonable that a 3rd harmonic existed and that it would contain an inscribed equilateral triangle where **w** is equal to $2\pi/3$. The general solution predicts theses values:

(w = 2π/3) 3rd harmonic $\quad$ **Ω ~= π/5.1043** $\quad$ **and** $\mathbf{R_{ta} = 2\sqrt{2}/\sqrt{3}}$

As I proceeded to set the compass to this conical radius, I realized that the conix would be located in the southern hemisphere. This made it difficult to draw. Thinking I was being clever, I thought I could use the other conical radius R_{ba} to draw the conix. However, when I used R_{ba} there was no geometric tiling taking place and the pattern of conix was not converging. So I had to revert back to using the conical radius R_{ta} which would be difficult to draw accurately with a classical compass. To resolve this limitation, I made a spherical compass (as seen colored green in Figure 15) that could reach into the southern hemisphere.

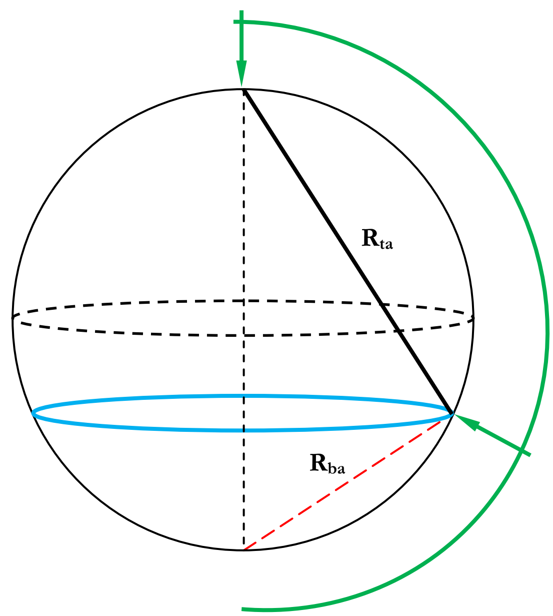


Figure 15

After a few iterations of tuning the compass to the precise length, I discovered the 3rd harmonic conix, as seen in Figure 16a, which is a view from the south pole of the sphere. Refer to Figure 16b where R_{ta} is used to mark off three chords and define an inscribed equilateral triangle in the conix.

I came to the following conclusion:

"***There is only one conix on a sphere where its conical radius R_{ta} is the same size as the side of its inscribed equilateral triangle"***.

I will refer to this conix as the 3rd harmonic conix.

I now know that the conical radius R_{ba} was too short to divide the conix into three equal arc segments. There is a unique geometric relationship between the conix and each of its conical radii. Just because the conical radius R_{ta} has a harmonic relationship with the sphere, this doesn't imply that conical radius R_{ba} does.

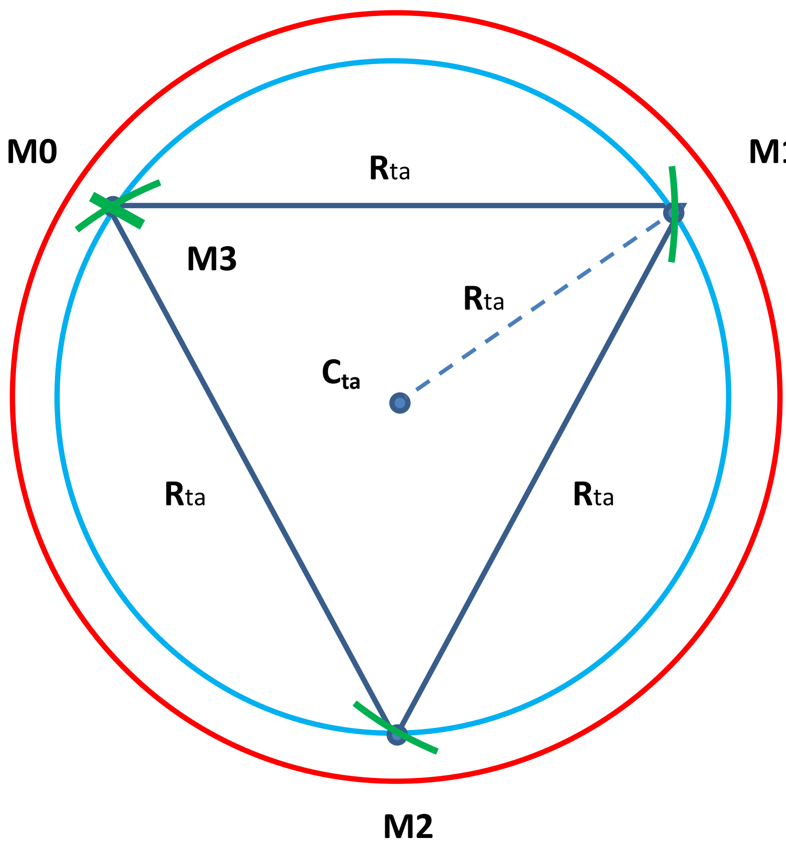


Figure 16a

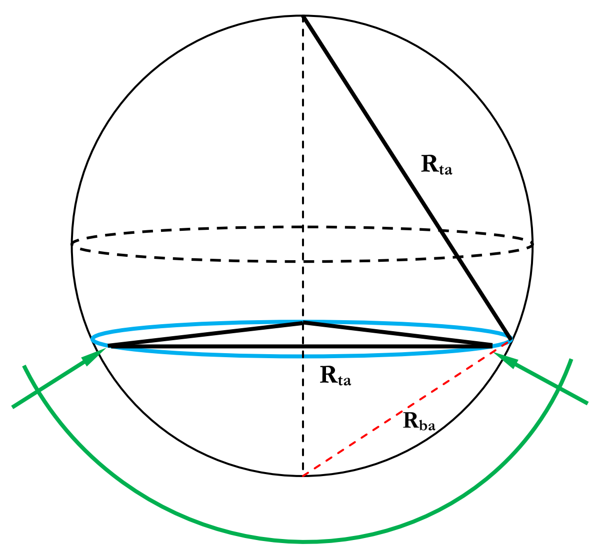


Figure 16b

As I had done with the 4th harmonic conix, I took the Flower of Life approach and started drawing more conix centered on each intersection point. After drawing four conix, the natural intersection of these harmonically coupled conix defined the vertices of a tetrahedron, which will be referred to as the 3rd harmonic note (See Figures 17-19.)

Figure 17

Figure 18

Figure 19

In addition to the bounding condition of the 6th harmonic, there were three fundamental geometric relationships discovered. What I find striking is that they were discovered purely with plain geometric construction techniques, using only a compass and sphere. (This simplicity could be ancient in origin!) Let's summarize what was discovered.

1) There is only one conix on a sphere where the length of the side of its inscribed pentagon has the same length of its conical radius R_{ta}. I will refer to this conix as the 5th harmonic conix.

    a. Twelve 5th harmonic conix self-intersect each other in a seamless and overlapping pattern and their intersection points define the vertices of an icosahedron and are referred to as the 5th harmonic note.

2) There is only one conix on a sphere where the length of the side of its inscribed square has the same length of its conical radius R_{ta}. I will refer to this conix as the 4th harmonic conix.

    a. Six 4th harmonic conix self-intersect each other in a seamless and overlapping pattern and their intersection points define the vertices of an octahedron and are referred to as the 4th harmonic note.

3) There is only one conix on a sphere where the length of the side of its inscribed equilateral triangle has the same length of its conical radius R_{ta}. I will refer to this conix as the 3rd harmonic conix.

    a. Four 3rd harmonic conix self-intersect each other in a seamless and overlapping pattern and their intersection points define the vertices of a tetrahedron and are referred to as the 3rd harmonic note.

These three shapes conform, interestingly, with Buckminster Fuller's three Primary Structures - Icosahedron, Octahedron and Tetrahedron!

What I hadn't explored yet, was if there were any harmonics beyond the 3rd harmonic. At first, it didn't seem feasible that there would be a 2nd harmonic conix because all of the Platonic solids were accounted for and given the trend of inscribed polygons, what would follow an inscribed triangle? There is no such thing as a two-sided polygon. But I was curious what the general solution would predict. The decreasing trend of chord angle results in **w** equal to $2\pi/2$ and the general solution predicts the following values for the 2nd harmonic conix:

(w = $2\pi/2$) 2nd harmonic $\Omega = \pi/6$ and $R_{ta} = \sqrt{3}$

Here are the steps I took to construct the 2nd harmonic note. The illustration in Figure 20 is a side view of the sphere. The compass is set with R_{ta} equal to $\sqrt{3}$. Then the compass is placed at an arbitrary point C_a and the first conix **X1** is drawn deep into the southern hemisphere. An arbitrary point on **X1** is defined as point C_b. The compass is then moved to C_b and the second conix **X2** is drawn refer to Figure 21. Notice how the second conix **X2** abuts the first conix **X1** at point C_c and that X1 is now divided into two equal arc segments by points C_b and C_c. The compass is then moved to C_c and the third conix **X3** is drawn which abuts both conix **X1** and **X2** as seen in Figure 22. Now each conix is divided in two by the abutment points with the other two conix.

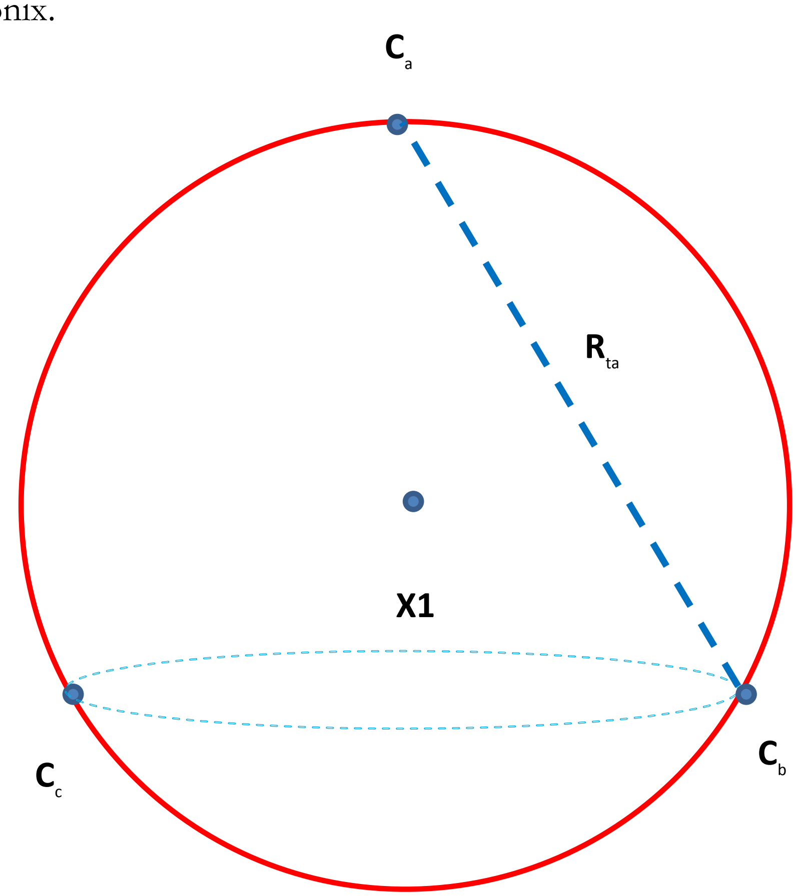


Figure 20

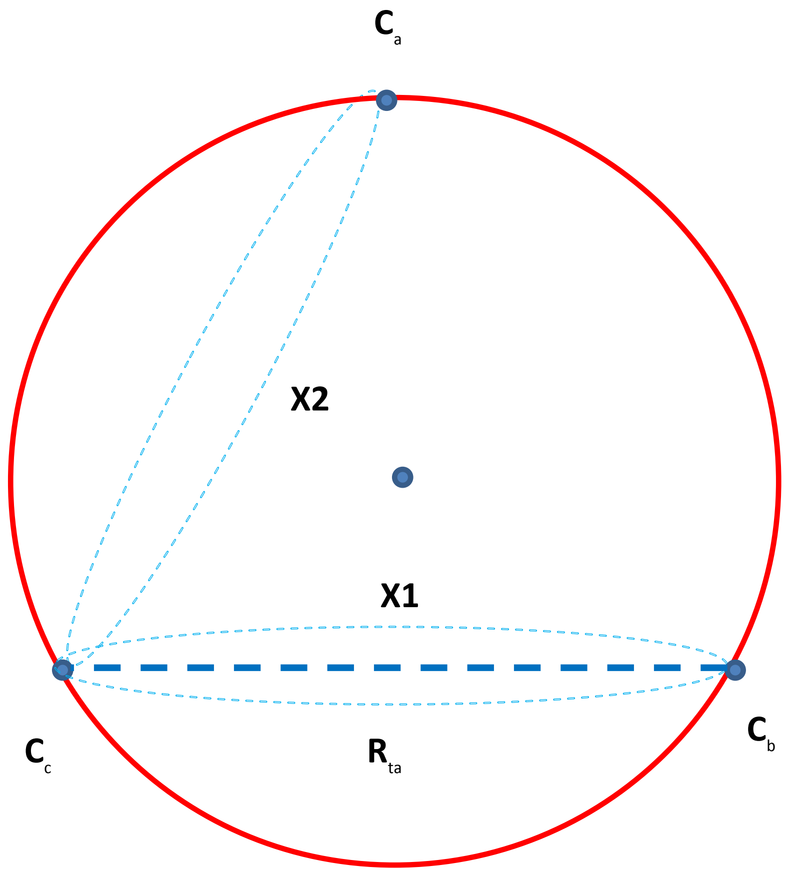


Figure 21

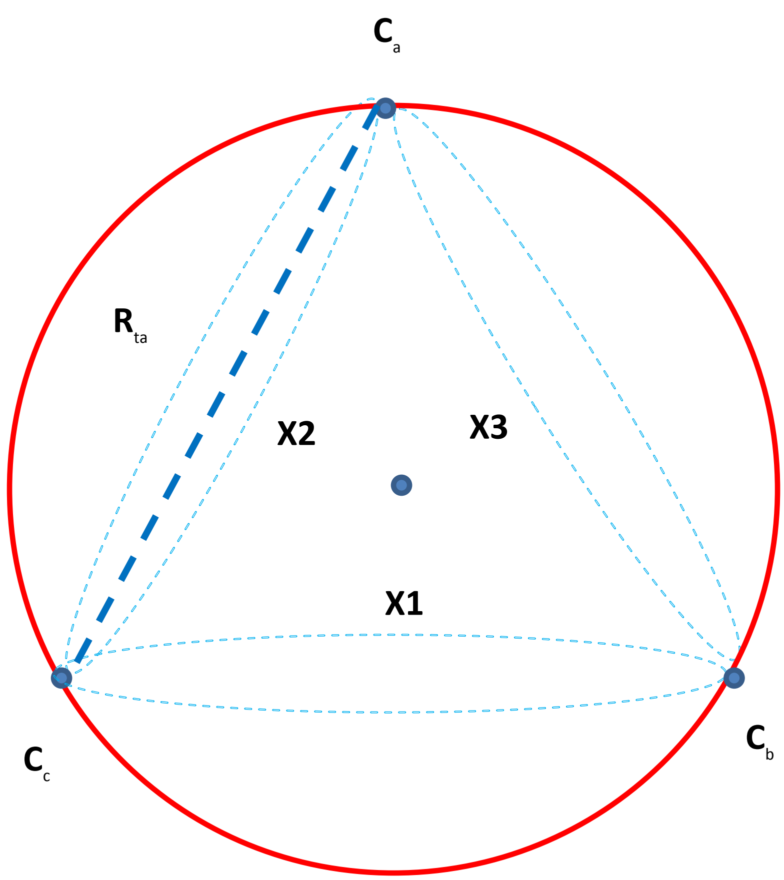


Figure 22

Refer to Figure 23-25 to see various perspective views of the 2nd harmonic note. Notice that three conix wrap around the circumference of the sphere and fully abut forming an inscribed equilateral triangle in the sphere's equator. It took me while to recognize that the 2nd harmonic note was associated with a polyhedron, even though it was not one of the Platonic solids. The 2nd harmonic note is based on a regular triangular prism.

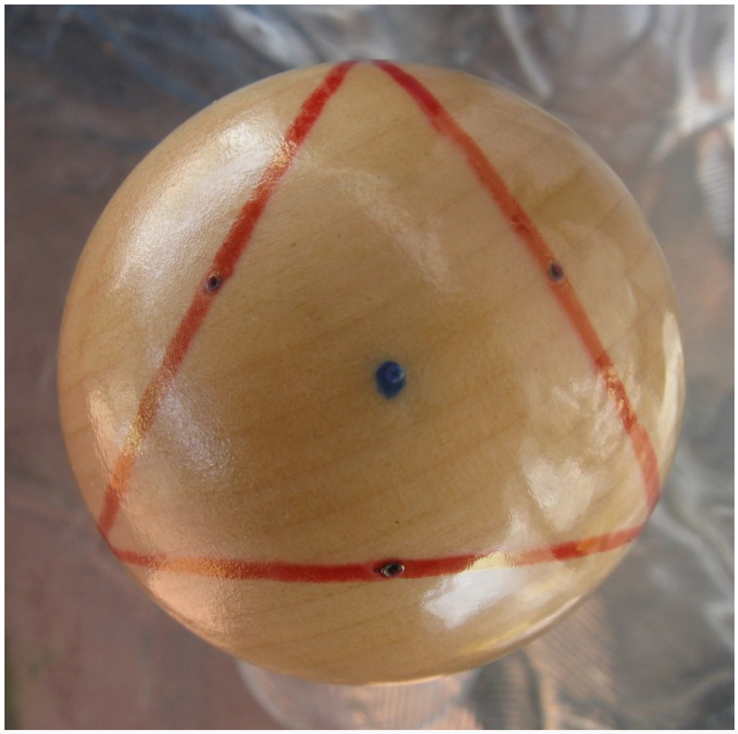
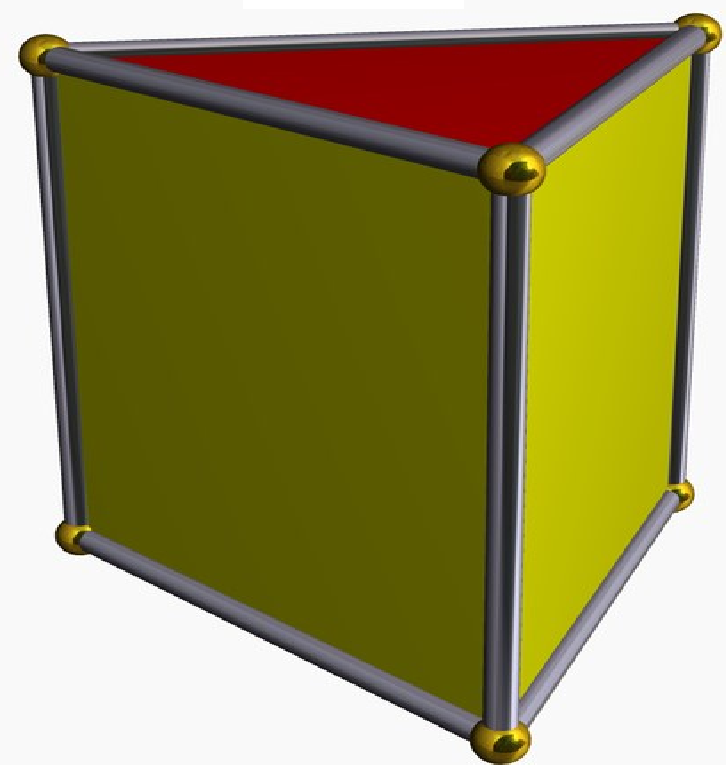


Figure 23



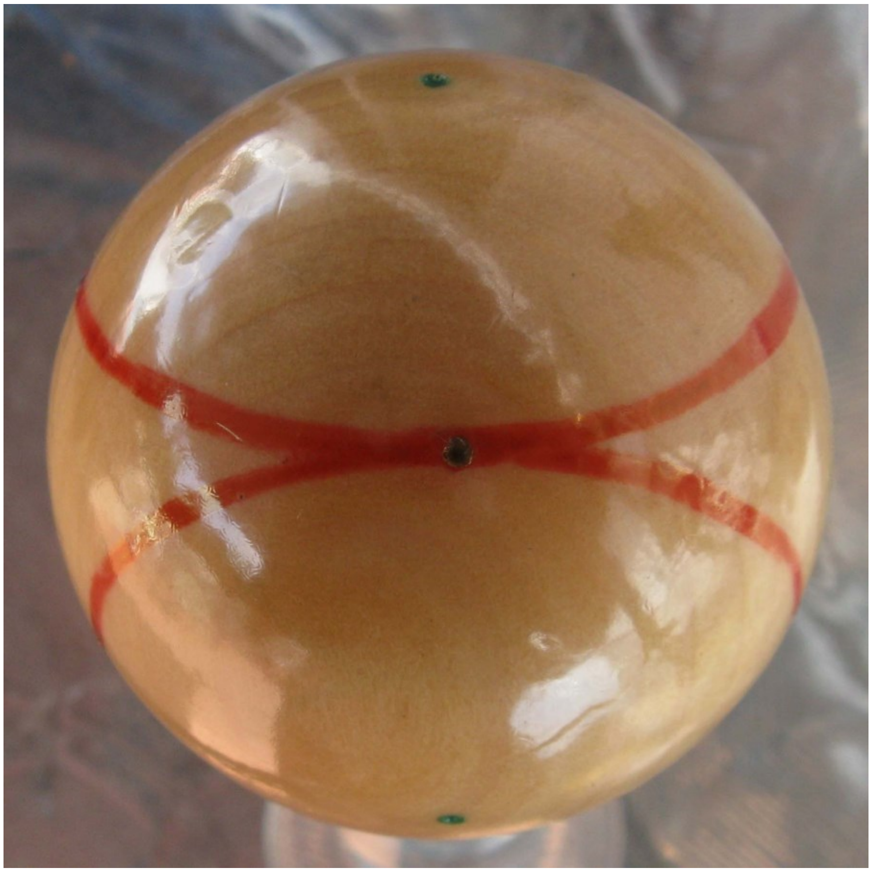


Figure 24

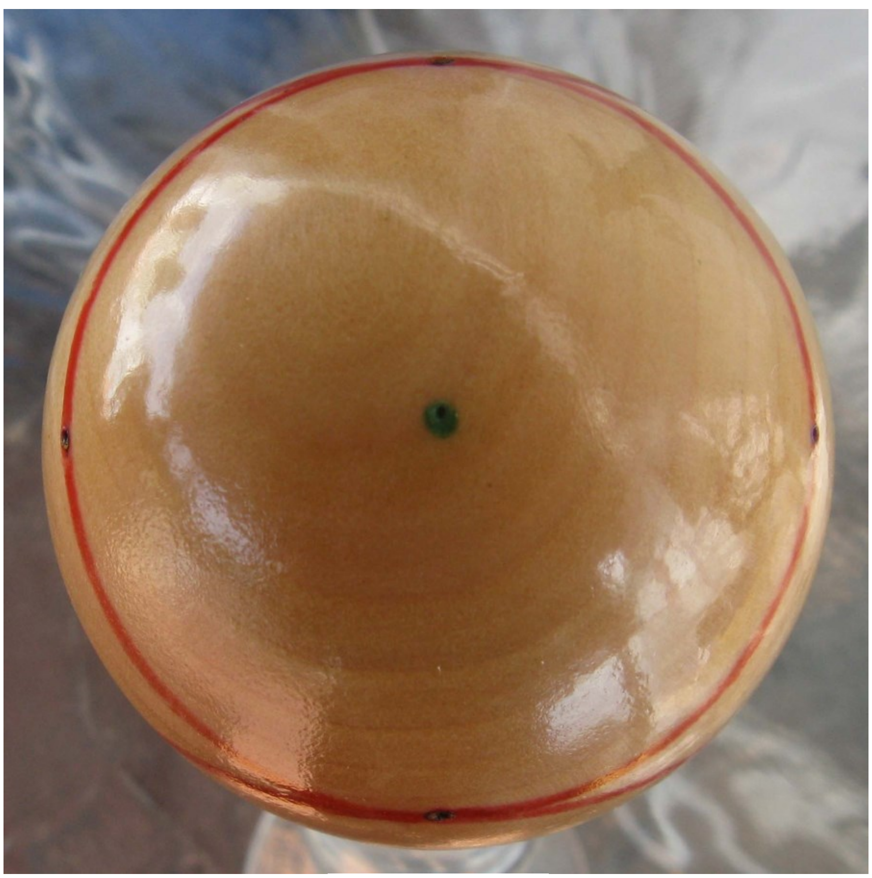


Figure 25

Chapter 3: Harmonic Notes Variations and Intervals

In this chapter I will explore the various harmonics in more detail. I have introduced the concept of a harmonic conix and a harmonic note. A harmonic conix has a unique relationship with the sphere it is constructed on. The conical radius of a conix (R_{ta} and R_{ba}) can be used to divide its circumference in equal arc segments. There are five fundamental harmonics (6th, 5th, 4th, 3rd and 2nd) and an infinite number of fractional harmonics. A harmonic note is a particular arrangement of harmonic conix on a sphere oriented around the axes of polyhedrons. It's not just any random arrangement of conix, it's a natural harmonic relationship of self-intersecting conix on a sphere. These arrangements of conix are self-guided by the intersection of the conix in a similar fashion as the "Flower of Life" described in Chapter 1. It just so happens that the three main harmonic notes are oriented around the Platonic solids.

The 5th harmonic defines the twelve vertices of an icosahedron.

The 4th harmonic defines the six vertices of an octahedron.

The 3rd harmonic defines the four vertices of a tetrahedron.

So let's explore the 5th harmonic note in more detail. The location of these vertices on the sphere fully defines the icosahedron, in that its edges are defined as the lines between two adjacent vertices and its faces are defined by the triangles form by three vertices. Refer to Figure 26 to see how one section of an icosahedron can be created by connecting the intersection points on the 5th harmonic note.

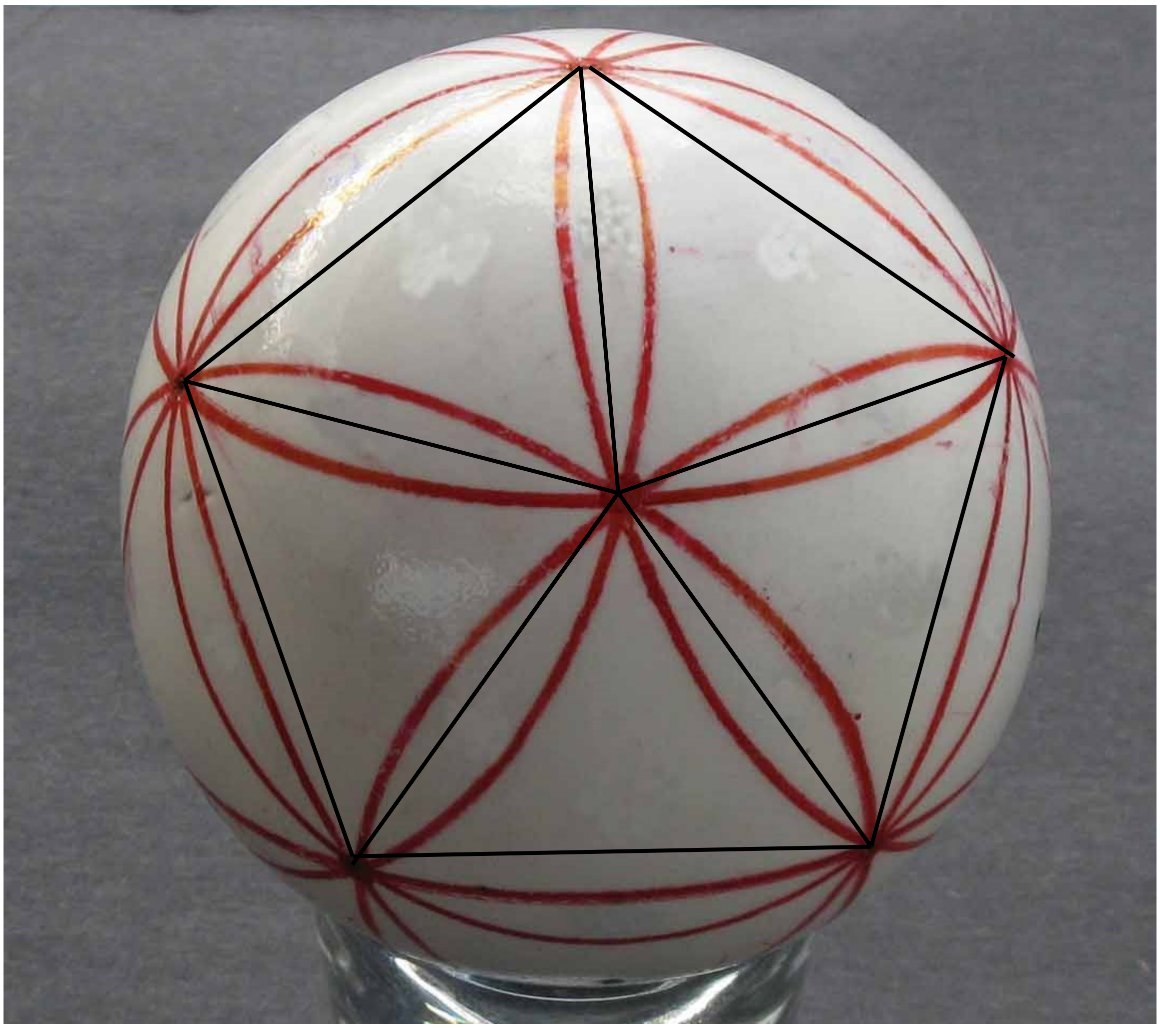


Figure 26

We know that the twelve vertices of an icosahedron also correspond to the twelve faces of its dual, the dodecahedron. However, as you can see in Figure 43a there are no intersection points that define the vertices of the pentagonal faces. There is a way around this. We can use a compass locate the twenty face axes of the icosahedron through triangulation from the three vertices that define each face, refer to Figure 27.

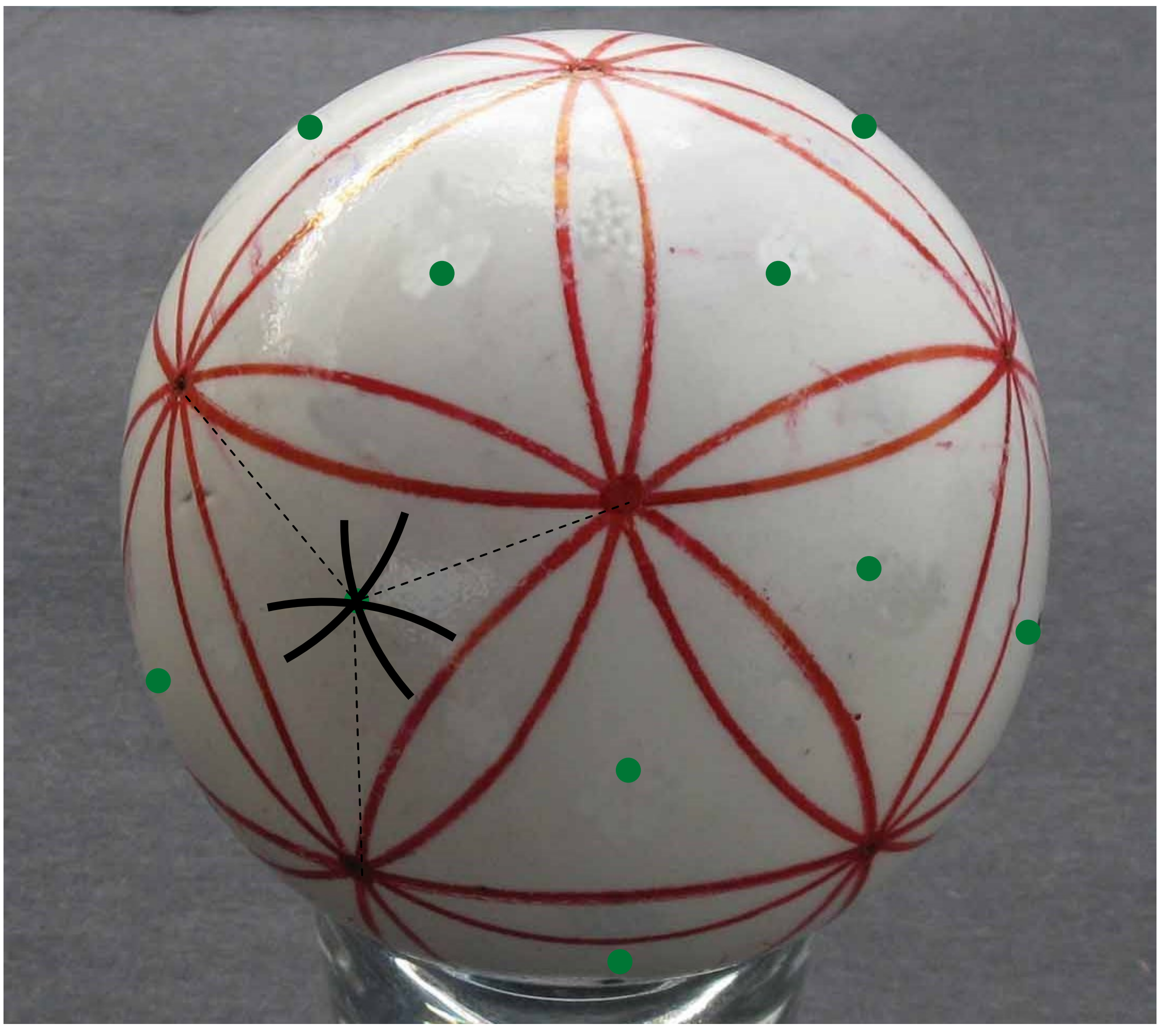


Figure 27

Now that the face axes are identified, the pentagonal face on the dodecahedron can be projected on the sphere's surface as seen in Figure 28.

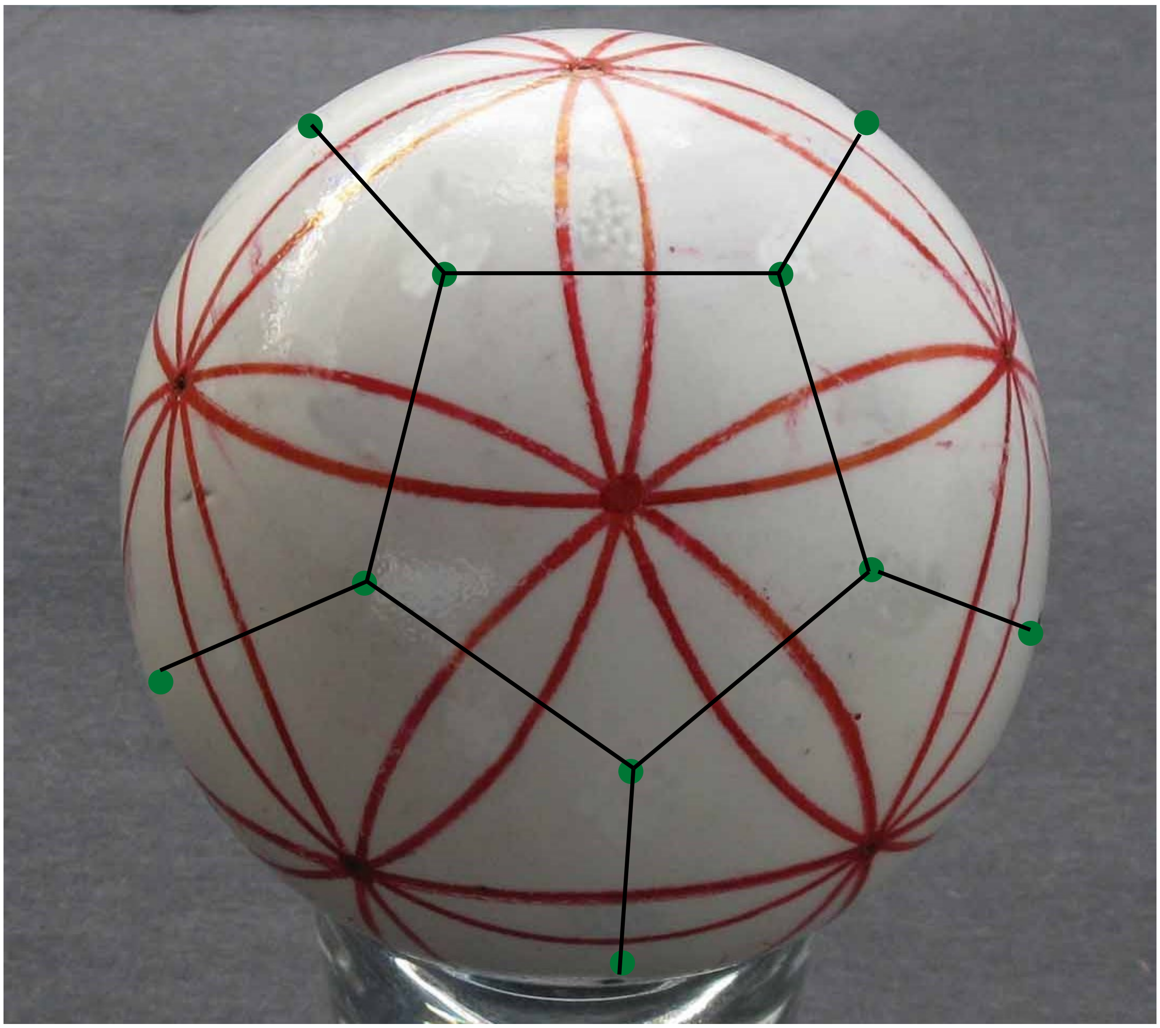


Figure 28

We can use a compass locate the thirty edge axes of the icosahedron through triangulation from the vertices and face axes, as seen in Figure 29.

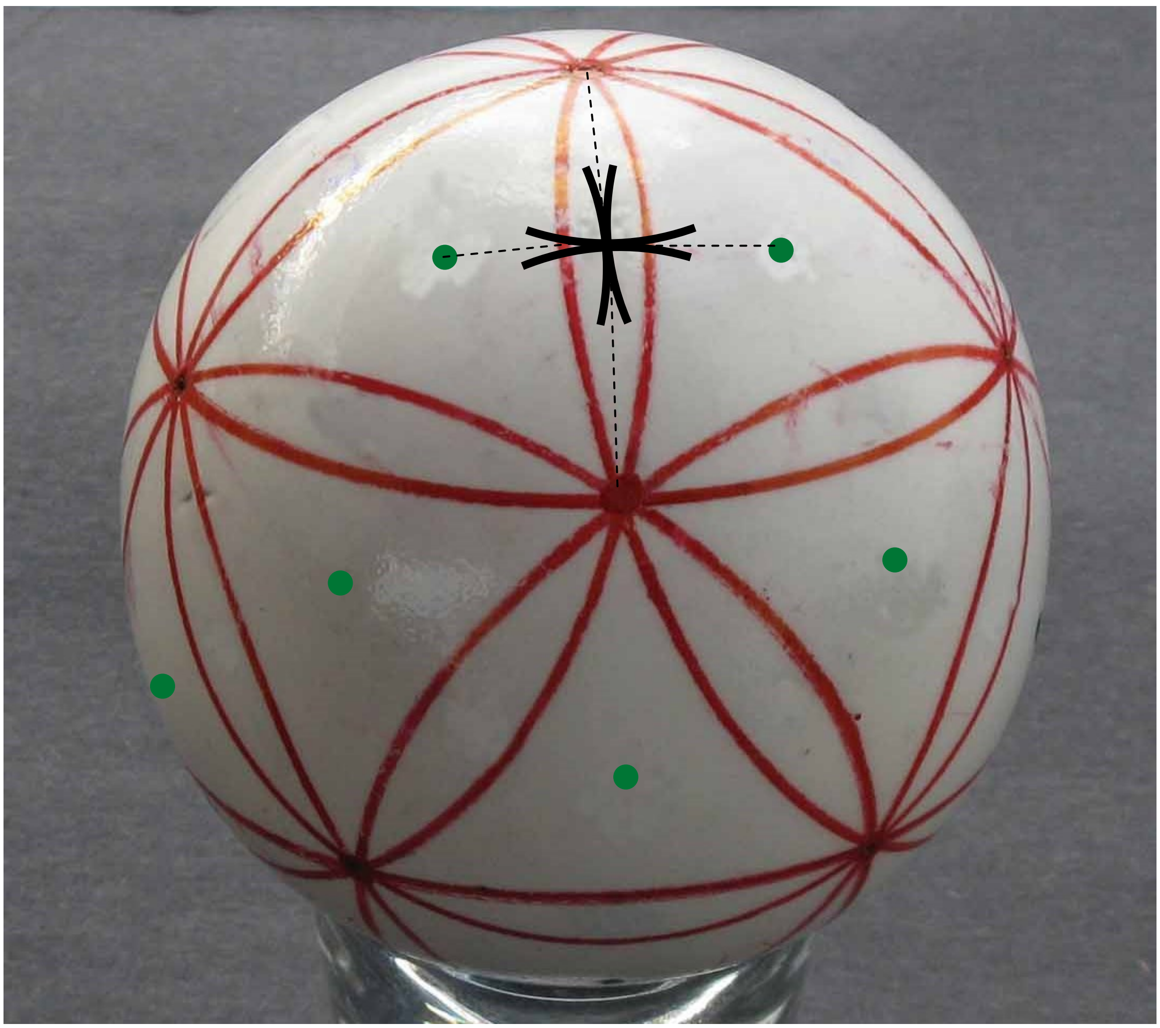


Figure 29

All sixty-two axes of the icosahedron can now be located as seen in Figure 30. The sphere is considered fully-pointed when all sixty two axes have been identified (twelve vertices, twenty faces and thirty edges). Fully pointed spheres can be defined in a similar way for the 4th and 3rd harmonic notes and are illustrated in Figures 31-32.

Figure 30

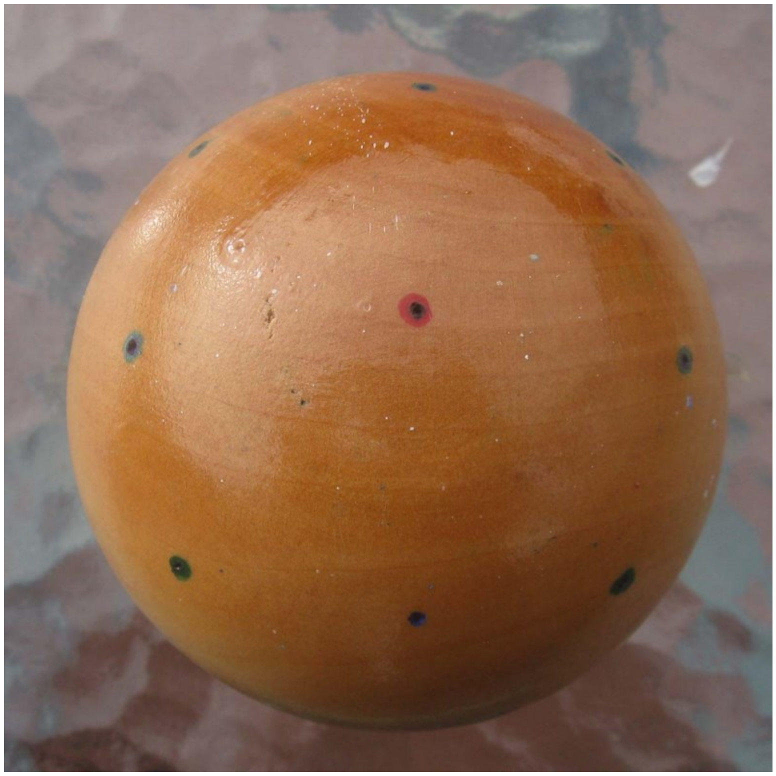


Figure 31

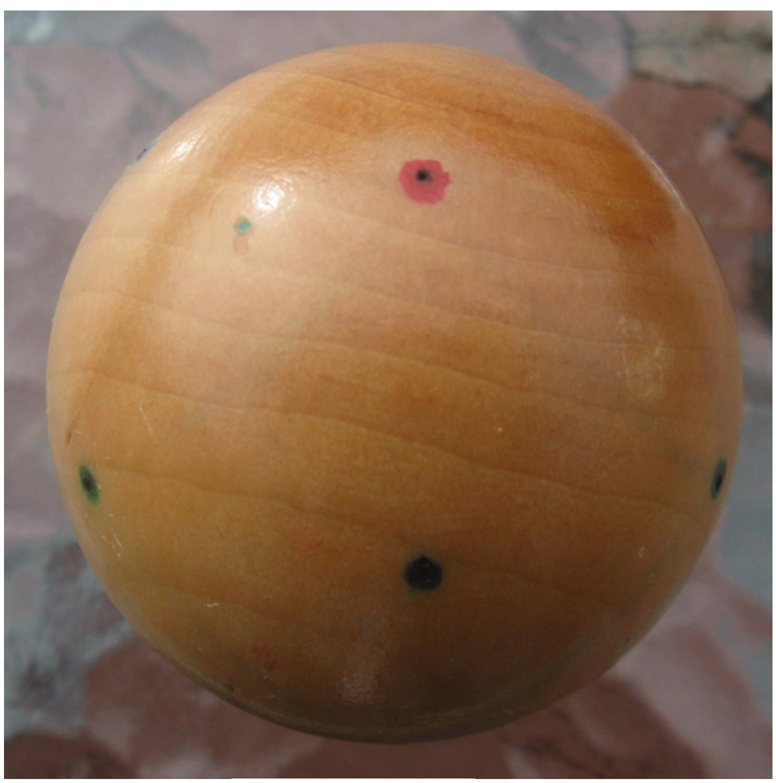


Figure 32

We have covered the three fundamental harmonic notes up to this point in the book. These fundamental notes are oriented around the vertices of its associated polyhedron and the conical radius R_{ta} of its conix is determined by the distance between adjacent vertices. The vertices are defined by the natural harmonic interaction between its conix. I have defined two other harmonic note variations which are oriented around the face and edge axes respectively. The notes oriented around the vertices are referred to as V-notes. The notes oriented around the face axis are referred to as F-notes. The notes oriented around the edge axis are referred to as E-notes.

There exists an interval of notes within each of these harmonic note variations (V-note, F-note and E-note). The conix in each harmonic note within an interval has a different conical radius R_{ta}. In the case of V-notes, the length of its conical radius is determined by the distance between the vertex and some other axis (vertex, face or edge). In the case of F-notes, the length of this conical radius is determined by the distance between the face axis and some other axis (vertex, face or edge). In the case of E-notes, the length of this conical radius is determined by the distance between the edge and some other axis (vertex, face or edge). In this way, every possible conix will be defined. This will be described in more detail in the following paragraphs.

5th harmonic note variations:

5V-note: Conix oriented to the vertices axes of the icosahedron.

The base 5V-note consists of twelve conix, each with a conical radius of the distance between adjacent vertices axes and oriented around the vertex axes.

5F-note: Conix oriented to the faces axes of the icosahedron.

The base 5F-note consists of twenty conix, each with a conical radius of the distance between adjacent face axes and oriented around the face axes.

5E-note: Conix oriented to the edges axes of the icosahedron.

The base 5E-note consists of thirty conix, each with a conical radius of the distance between adjacent edge axes and oriented around the edge axes.

4th harmonic note variations:

4V-note: Conix oriented to the vertices axes of the octahedron.

The base 4V-note consists of six conix, each with a conical radius of the distance between adjacent vertices axes and oriented around the vertex axes.

4F-note: Conix oriented to the faces axes of the octahedron.

The base 4F-note consists of eight conix, each with a conical radius of the distance between adjacent face axes and oriented around the face axes.

4E-note: Conix oriented to the edges axes of the octahedron.

The base 4E-note consists of twelve conix, each with a conical radius of the distance between adjacent edge axes and oriented around the edge axes.

3rd harmonic note variations:

3V-note: Conix oriented to the vertices axes of the tetrahedron.

The base 3V-note consists of four conix, each with a conical radius of the distance between adjacent vertices axes and oriented around the vertex axes.

3F-note: Conix oriented to the faces axes of the tetrahedron.

The base 3F-note consists of four conix, each with a conical radius of the distance between adjacent face axes and oriented around the face axes.

3E-note: Conix oriented to the edges axes of the tetrahedron.

The base 3E-note consists of six conix, each with a conical radius of the distance between adjacent edge axes and oriented around the edge axes.

Given that there are numerous harmonic notes, I developed a nomenclature to classify them. Each harmonic has three variations or intervals of notes. Let me describe this nomenclature using the 5th harmonic as an example.

This is the nomenclature to identify each conix.

$$\boldsymbol{HI}_{na}^{na}$$

Where *"$\boldsymbol{H}$"* indicates the harmonic [$\boldsymbol{2{:}5}$] and *"$\boldsymbol{I}$"* indicates the interval [$\boldsymbol{V}|\boldsymbol{F}|\boldsymbol{E}$]. The *upper* "$\boldsymbol{n}$" [0:3] indicates that the radius is smaller than the base conix and the larger the value, the smaller the radius. The *lower* "$\boldsymbol{n}$" [0:6] indicates that the radius is larger than the base conix and the larger the value, the larger the radius. The *"$\boldsymbol{a}$"* [$v|f|e$] indicates which axis is intersected by the conix and is meant to assist in visualization and construction. Here are some examples for the 5th harmonic variations.

5V-interval:

$\mathbf{5}\boldsymbol{V}$ Describes the tonic or base note created by a conix with a radius determined by the distance between adjacent vertex axes of the icosahedron.

$\mathbf{5}\boldsymbol{V}_{1f}$ Describes the note created by a conix with a radius larger than the base note and the conix intersects the next furthest axis (face) of the icosahedron.

$\mathbf{5}\boldsymbol{V}^{1e}$ Describes the note created by a conix with a radius smaller than the base note and the conix intersects the next furthest axis (edge) of the icosahedron.

5F-interval:

$\mathbf{5}\boldsymbol{F}$ Describes the tonic or base note created by a conix with a radius determined by the distance between adjacent face axes of the icosahedron.

$\mathbf{5}\boldsymbol{F}_{1e}$ Describes the note created by a conix with a radius larger than the base note and the conix intersects the next furthest axis (edge) of the icosahedron.

$\mathbf{5}\boldsymbol{F}^{1v}$ Describes the note created by a conix with a radius smaller than the base note and the conix intersects the next furthest axis (vertex) of the icosahedron.

5E-interval:

$\mathbf{5}\boldsymbol{E}$ Describes the tonic or base note created by a conix with a radius determined by the distance between adjacent edge axes of the icosahedron.

$\mathbf{5}\boldsymbol{E}_{1f}$ Describes the note created by a conix with a radius larger than the base note and the conix intersects the next furthest axis (face) of the icosahedron.

$\mathbf{5}\boldsymbol{E}^{1v}$ Describes the note created by a conix with a radius smaller than the base note and the conix intersects the next furthest axis (vertex) of the icosahedron.

The following Figures 33-35 are used to assist in identifying the different conix for each interval. These illustrations are a two dimensional projection of an icosahedron and viewed from the different axes. The various colored line segments represent a two dimensional projection of the conical radius R_{ta} and the two axes which they connect to.

Figure 33 is a vertex view of the icosahedron and the center of each conix is located at the vertex in the center of the illustration. The smallest of the 5th harmonic V-notes is the $\mathbf{5V^{3e}}$ conix and the length of its conical radius R_{ta} is determined by the distance between a vertex and the closest edge axis. The largest of the 5th harmonic V-notes is the $\mathbf{5V_{2e}}$ conix and the length of its conical radius R_{ta} is determined by the distance between a vertex and the farthest edge axis without going into the southern hemisphere.

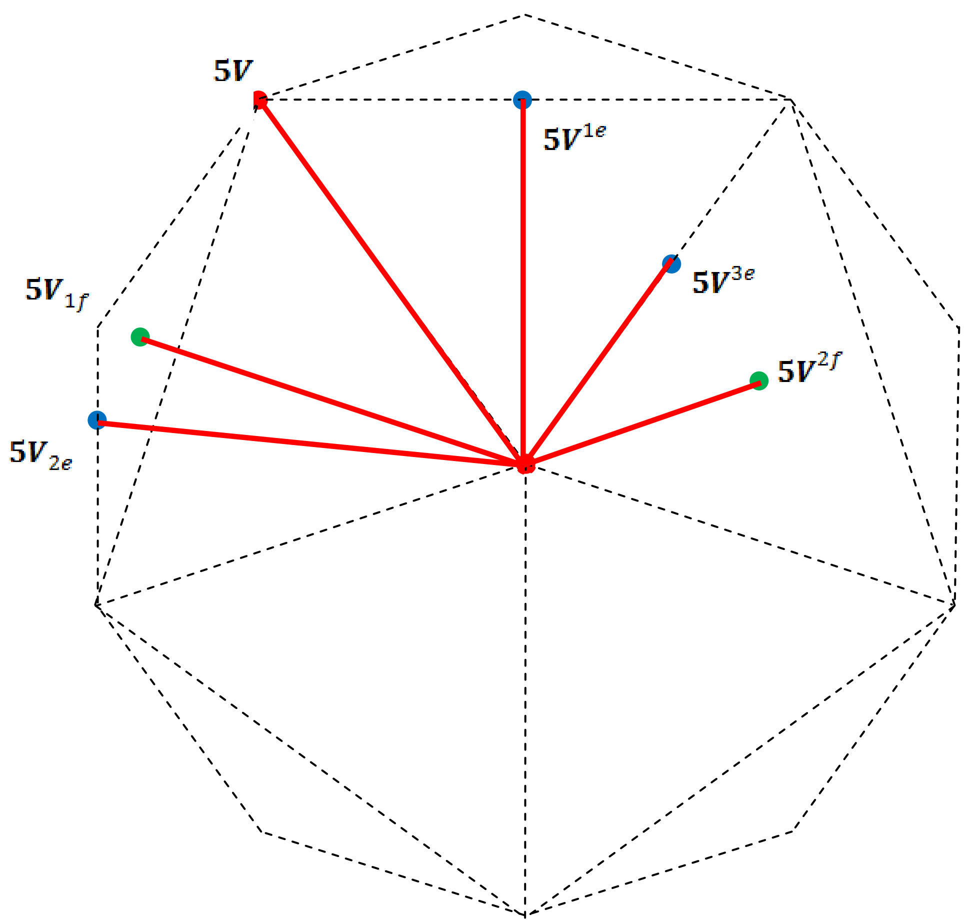


Figure 33

Figure 34 is a face view of the icosahedron and the center of each conix is located at the face axis located at the center of the illustration. The smallest of the 5th harmonic F-notes is the $\mathbf{5F^{2e}}$ conix and the length of its conical radius R_{ta} is determined by the distance between a face axis and the closest edge axis. The largest of the 5th harmonic F-notes is the $\mathbf{5F_{5e}}$ conix and the length of its conical radius R_{ta} is determined by the distance between a face axis and the farthest edge axis without going into the southern hemisphere.

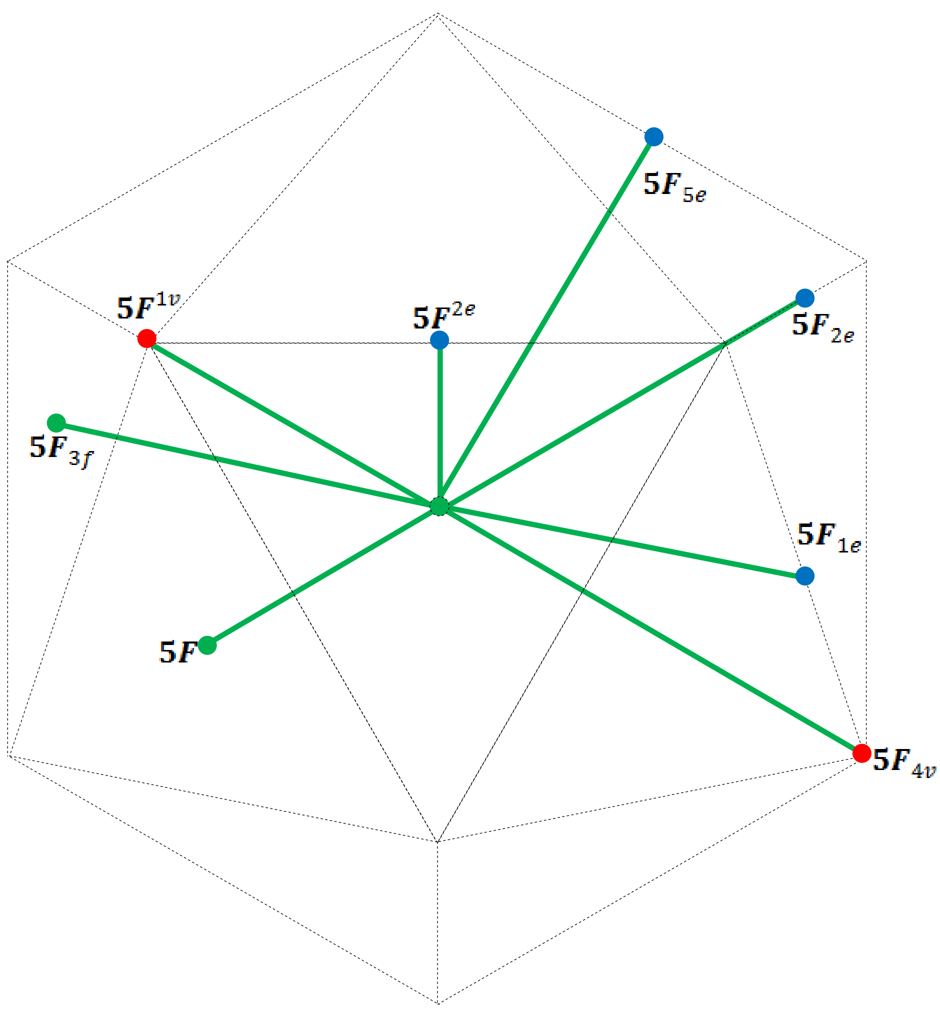


Figure 34

Figure 35 is an edge view of the icosahedron and the center of each conix is located at the edge axis located at the center of the illustration. The smallest of the 5th harmonic E-notes is the **5E^{2f}** conix and the length of its conical radius R_{ta} is determined by the distance between an edge axis and the closest face axis. The largest of the 5th harmonic E-notes is the **5E$_{6e}$** conix and the length of its conical radius R_{ta} is determined by the distance between an edge axis and the farthest edge axis without going into the southern hemisphere.

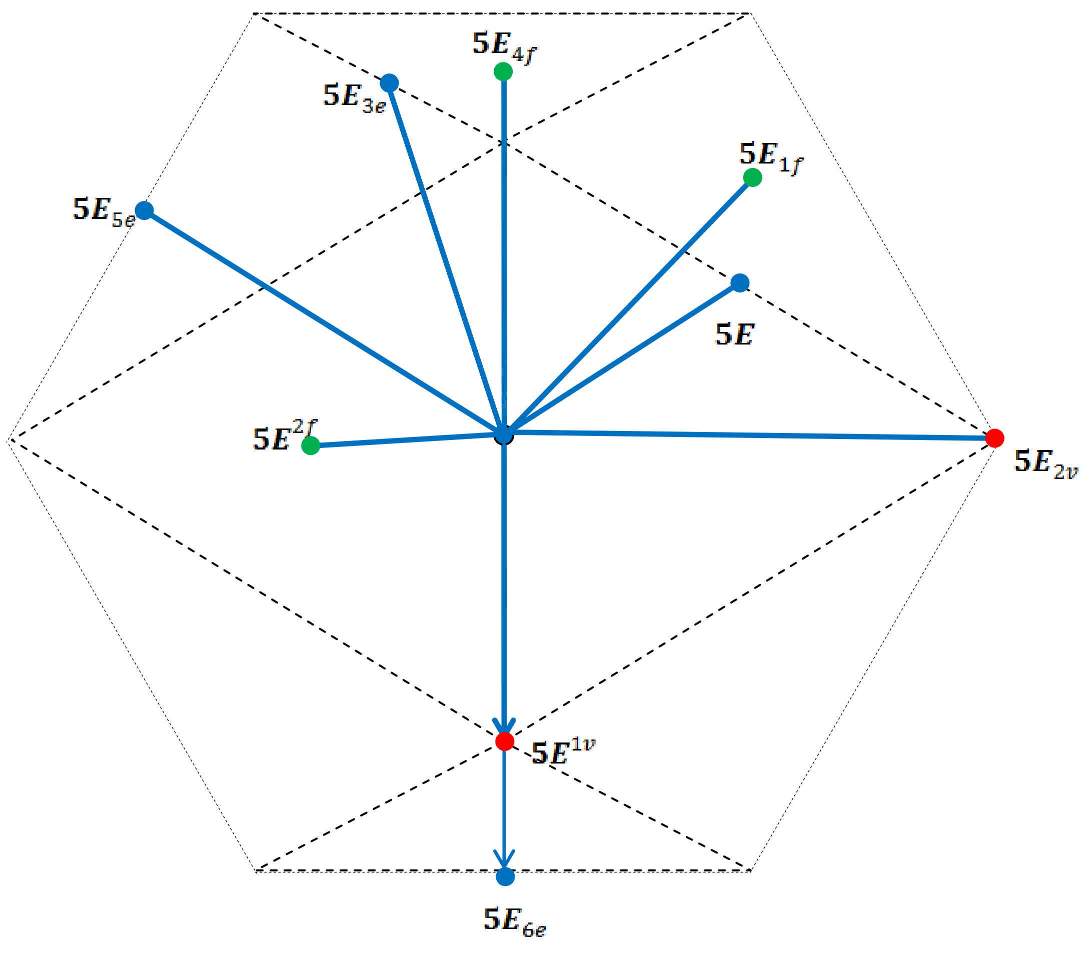


Figure 35

Using Figures 33-35, I was able to identify all of the possible conix of each of the harmonic notes for each interval (V-note, F-note and E-note) of the 5th harmonic. I also listed all of the conix for the other harmonic notes as well, see Table I.

Table I

V- notes	F-notes	E- notes
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$
$5V_{1f}$	$5F_{4v}$	$5E_{5e}$
$5V$	$5F_{3f}$	$5E_{4f}$
$5V^{1e}$	$5F_{2e}$	$5E_{3e}$
$5V^{2f}$	$5F_{1e}$	$5E_{2v}$
$5V^{3e}$	$5F$	$5E_{1f}$
	$5F^{1v}$	$5E$
	$5F^{2e}$	$5E^{1v}$
		$5E^{2f}$
	$4F_{1e}$	$4E_{1v}$
$4V$	$4F$	$4E$
$4V^{1f}$	$4F^{1v}$	$4E^{1v}$
$4V^{2e}$	$4F^{2e}$	$4E^{2f}$
$3V_{1e}$	$3V_{1e}$	$3V_{1e}$
$3V$	$3V$	$3V$
$3V^{1e}$	$3V^{1e}$	$3V^{1e}$
$2V_{2f}$		
$2V_{1v}$		
$2V$		

There are twenty-three conix within the 5th harmonic intervals, six within the 5V-interval, eight within the 5F-interval and nine within the 5E-interval. My first observation was that the sphere's equator was the largest conix and was common to all three intervals. Secondly, some conix were common to two intervals and thirdly, some conix were unique to one interval. Refer to Table II for a grouped list in descending order of the length of the conical radius for the 5th harmonic. I simply used a compass to establish the order.

Table II

V- notes	F-notes	E- notes
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$
$5V_{1f}$	$5F_{4v}$	
		$5E_{5e}$
	$5F_{3f}$	
	$5F_{2e}$	$5E_{4f}$
5V		
		$5E_{3e}$
$5V^{1e}$		$5E_{2v}$
	$5F_{1e}$	$5E_{1f}$
	5F	
$5V^{2f}$	$5F^{1v}$	
		5E
$5V^{3e}$		$5E^{1v}$
	$5F^{2e}$	$5E^{2f}$

By measuring the various conical radii of each conix for all of the 5th harmonic notes I was able to establish the following relationship between conix. These are the basic harmonic conix identities. There is nothing overly significant about these identities that the reader needs to be immediately concerned about. In some cases the same size of conix is used in two different notes but centered on different axes. They all can be easily visualized by using a three dimensional model of an icosahedron and highlighting each radius on its surface. As a result, this will minimize that number of geometric solutions required to derive the precise value of each radii.

$$\mathbf{5}\boldsymbol{F}_{1e} = \mathbf{5}\boldsymbol{E}_{1f}$$

$$\mathbf{5}\boldsymbol{V}^{2f} = \mathbf{5}\boldsymbol{F}^{1v}$$

$$\mathbf{5}\boldsymbol{V}^{3e} = \mathbf{5}\boldsymbol{E}^{1v}$$

$$\mathbf{5}\boldsymbol{F}^{2e} = \mathbf{5}\boldsymbol{E}^{2f}$$

$$\mathbf{5}\boldsymbol{V}^{1e} = \mathbf{5}\boldsymbol{E}_{2v} = \mathbf{5}\boldsymbol{V}^{2f} + \mathbf{5}\boldsymbol{E}^{2f}$$

$$\mathbf{5}\boldsymbol{V}^{1e} = \mathbf{5}\boldsymbol{E}_{2v} = \mathbf{5}\boldsymbol{V}^{2f} + \mathbf{5}\boldsymbol{F}^{2e}$$

$$\mathbf{5}\boldsymbol{V}^{1e} = \mathbf{5}\boldsymbol{E}_{2v} = \mathbf{5}\boldsymbol{F}^{1v} + \mathbf{5}\boldsymbol{E}^{2f}$$

$$\mathbf{5}\boldsymbol{V}^{1e} = \mathbf{5}\boldsymbol{E}_{2v} = \mathbf{5}\boldsymbol{F}^{1v} + \mathbf{5}\boldsymbol{F}^{2e}$$

$$\mathbf{5}\boldsymbol{V}_{1f} = \mathbf{5}\boldsymbol{F}_{4v} = \mathbf{5}\boldsymbol{V}^{2f} + 2 * \mathbf{5}\boldsymbol{E}^{2f}$$

$$\mathbf{5}\boldsymbol{V}_{1f} = \mathbf{5}\boldsymbol{F}_{4v} = \mathbf{5}\boldsymbol{V}^{2f} + 2 * \mathbf{5}\boldsymbol{F}^{2e}$$

$$\mathbf{5}\boldsymbol{V}_{1f} = \mathbf{5}\boldsymbol{F}_{4v} = \mathbf{5}\boldsymbol{F}^{1v} + 2 * \mathbf{5}\boldsymbol{E}^{2f}$$

$$\mathbf{5}\boldsymbol{V}_{1f} = \mathbf{5}\boldsymbol{F}_{4v} = \mathbf{5}\boldsymbol{F}^{1v} + 2 * \mathbf{5}\boldsymbol{F}^{2e}$$

$$\mathbf{5}\boldsymbol{V}_{1f} = \mathbf{5}\boldsymbol{F}_{4v} = \mathbf{5}\boldsymbol{V}^{1e} + \mathbf{5}\boldsymbol{E}^{2f}$$

$$5V_{1f} = 5F_{4v} = 5V^{1e} + 5F^{2e}$$

$$5V_{1f} = 5F_{4v} = 5E_{2v} + 5E^{2f}$$

$$5V_{1f} = 5F_{4v} = 5E_{2v} + 5F^{2e}$$

$$5F_{2e} = 5E_{4f} = 5V^{3e} + 5F^{1v}$$

$$5F_{2e} = 5E_{4f} = 5V^{3e} + 5V^{2f}$$

$$5F_{2e} = 5E_{4f} = 5E^{1v} + 5F^{1v}$$

$$5F_{2e} = 5E_{4f} = 5E^{1v} + 5V^{2f}$$

$$5V_{2e} = 5F_{5e} = 5E_{6e} = 5E^{1v} + 5E_{2v}$$

$$5F = 2 * 5E^{2f}$$

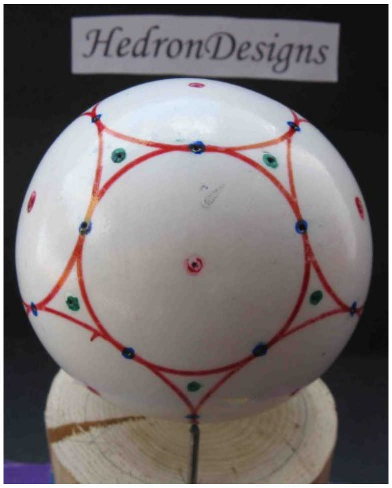
$$5F = 2 * 5F^{2e}$$

$$5V = 2 * 5V^{3e}$$

$$5V = 2 * 5E^{1v}$$

I have essentially defined all possible harmonic conix that can be used for every possible harmonic note. I have constructed each harmonic note and they are illustrated on the following pages. Each row of illustrations contains the three views (vertex, face and edge) of each harmonic note.

Vertex view **Face view** **Edge view**

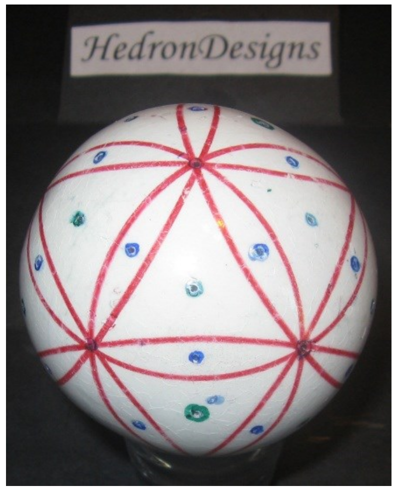


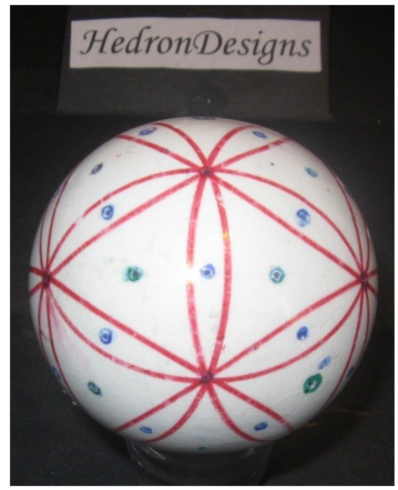
$5V^{3e}$

$5V^{2f}$

$5V^{1e}$

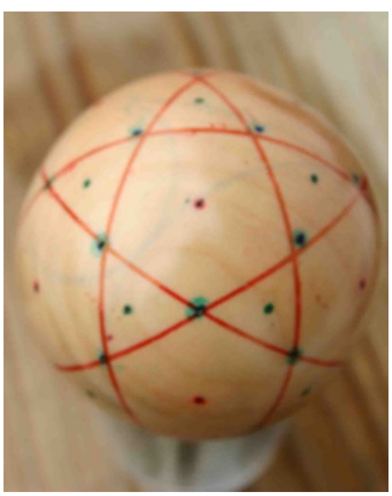
Vertex view **Face view** **Edge view**





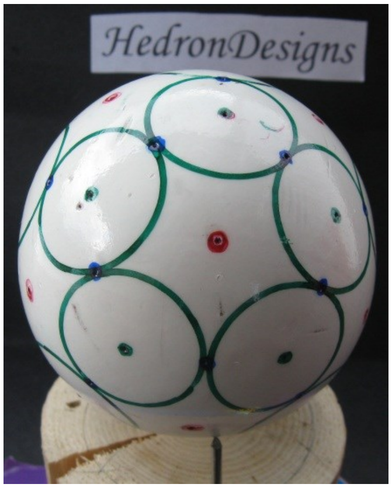
5V

$5V_{1f}$



$5V_{2e}$

Vertex view **Face view** **Edge view**



5F^{2e}

5F^{1v}

5F

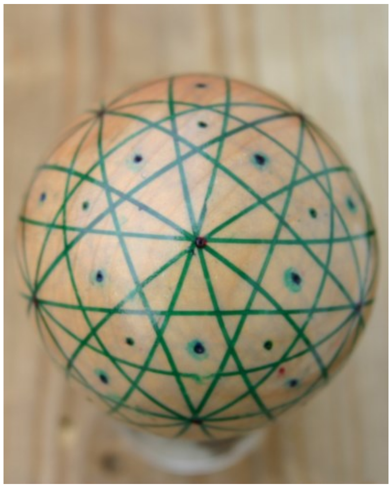
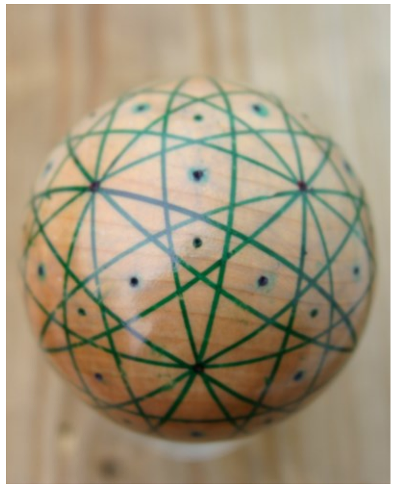
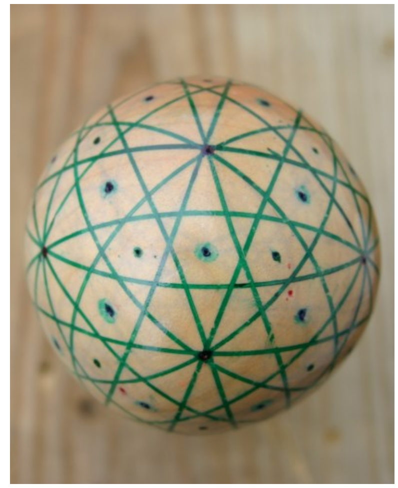
Vertex view **Face view** **Edge view**

$5F_{1e}$

$5F_{2e}$

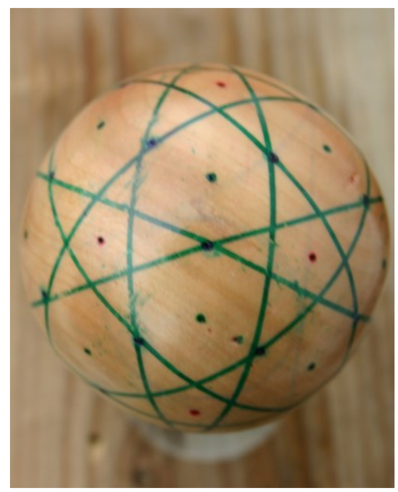
$5F_{3f}$

Vertex view **Face view** **Edge view**

5F$_{4v}$





5F$_{5e}$

Vertex view **Face view** **Edge view**

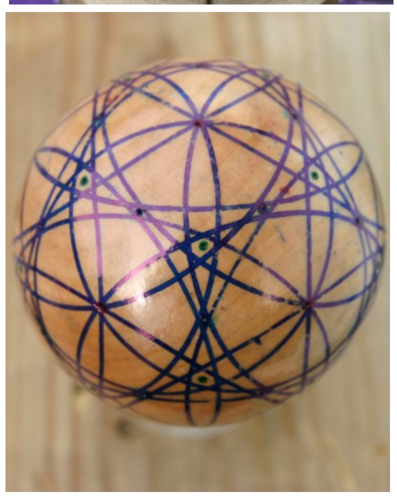
$5E^{2v}$

$5E^{1v}$

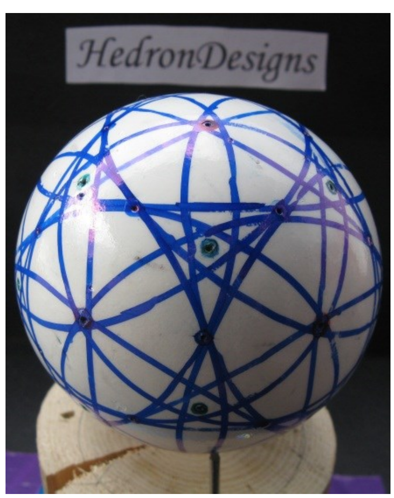
5E

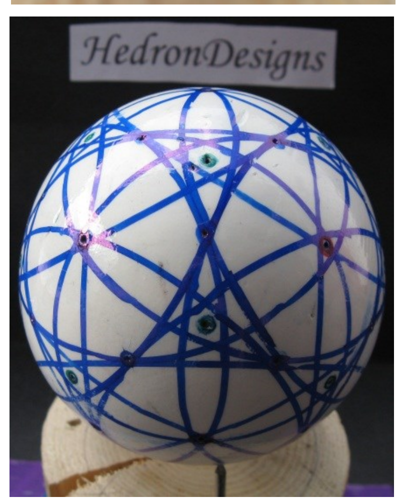
Vertex view **Face view** **Edge view**

$5E_{1f}$

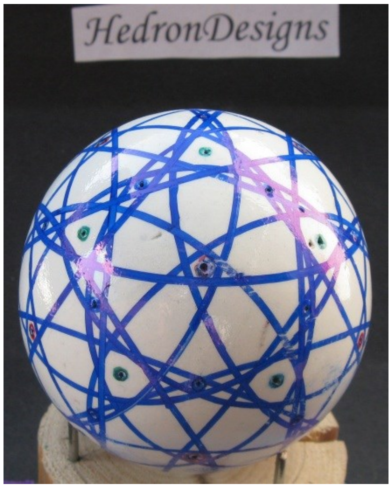

$5E_{2v}$

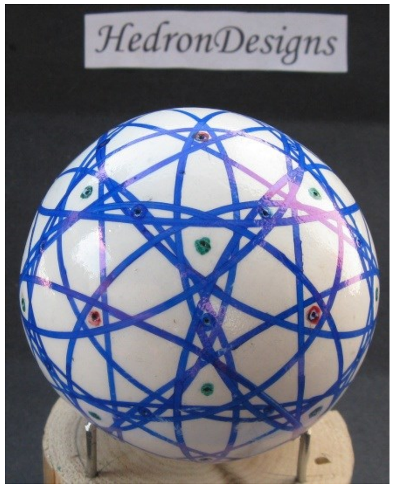


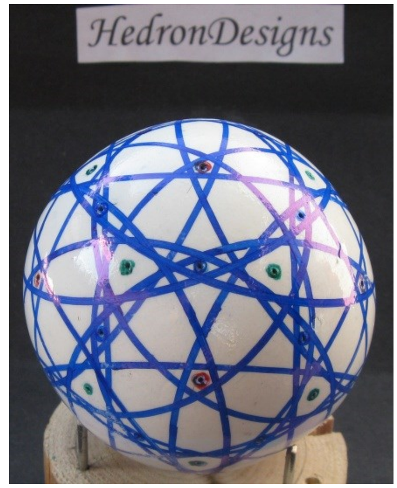


$5E_{3e}$

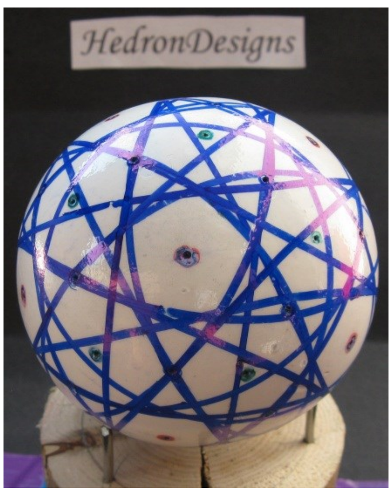
Vertex view **Face view** **Edge view**

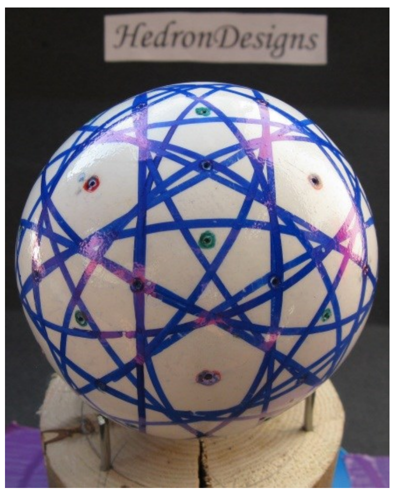


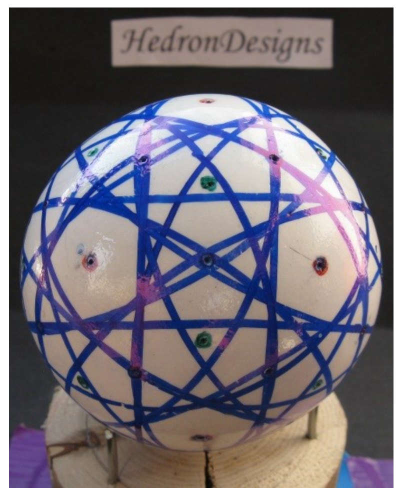




$5E_{4f}$





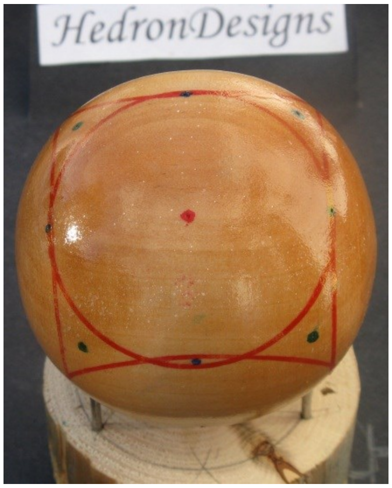


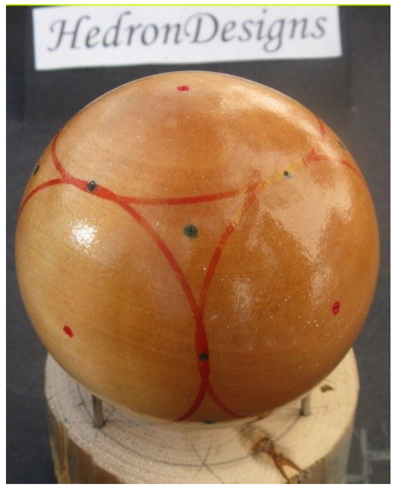
$5E_{5e}$

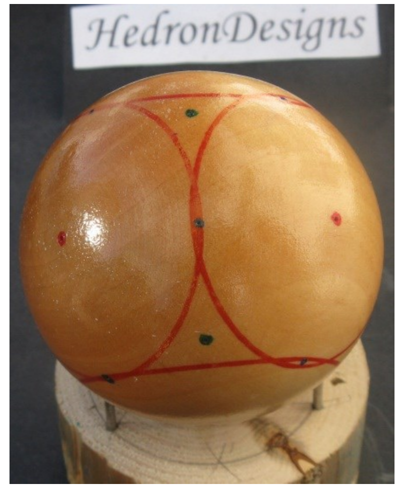


$5E_{6e}$

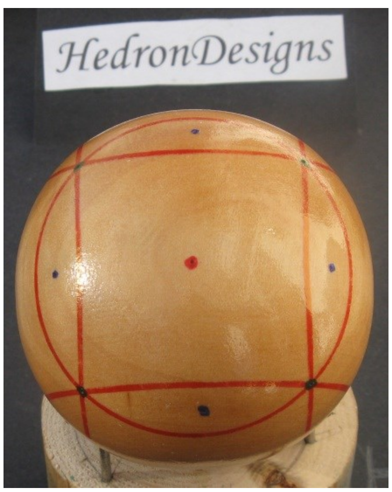
Vertex view **Face view** **Edge view**



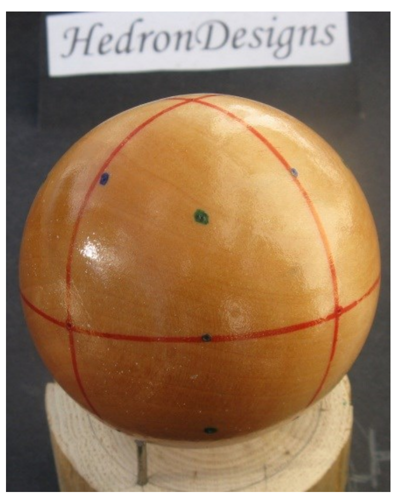




$4V^{2e}$



$4V^{1f}$

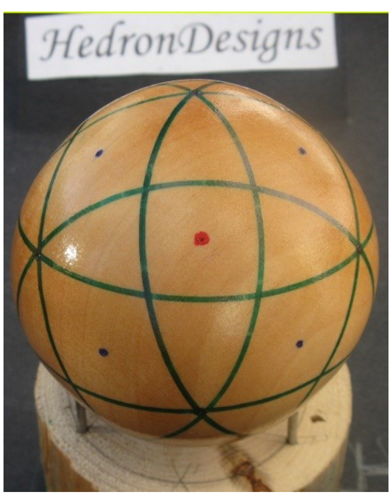


4V

Vertex view **Face view** **Edge view**

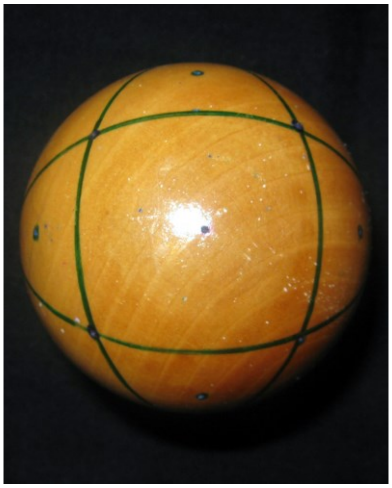
$4F^{2e}$

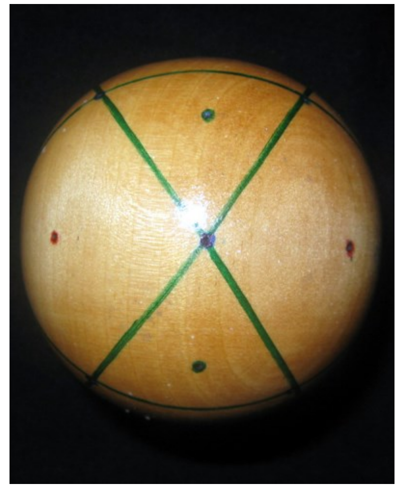
$4F^{1v}$



4F

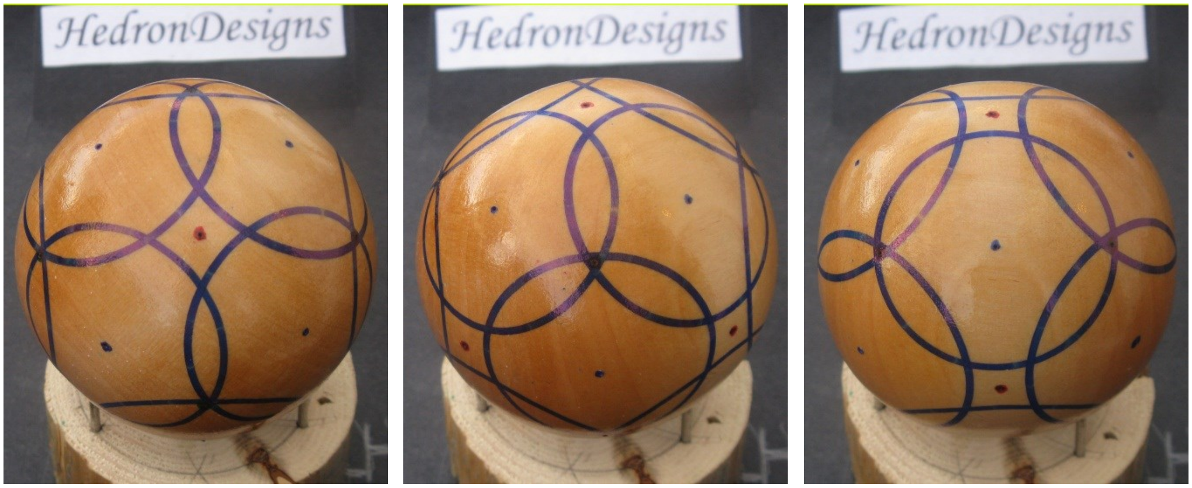
Vertex view **Face view** **Edge view**



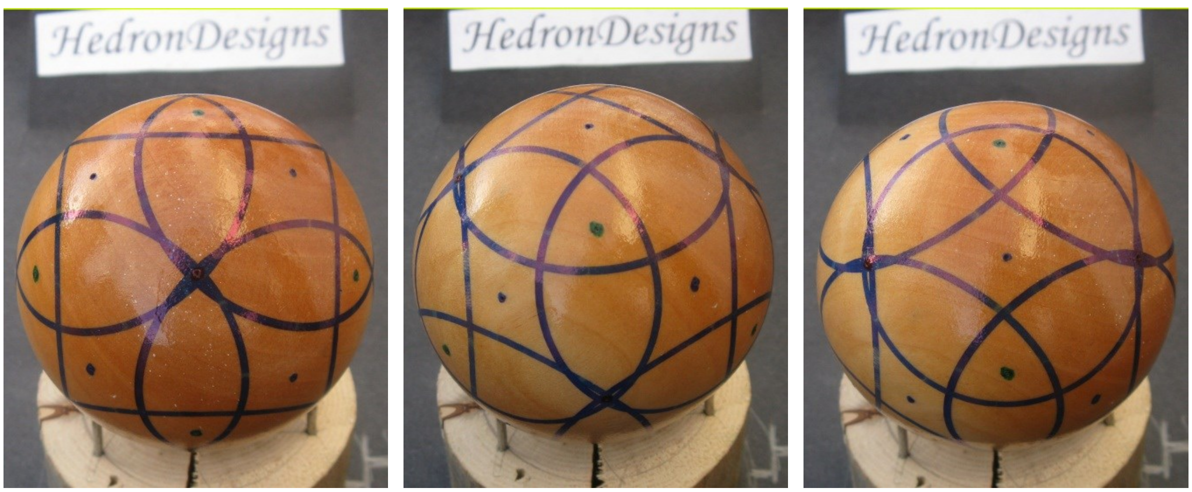


$4F_{1e}$

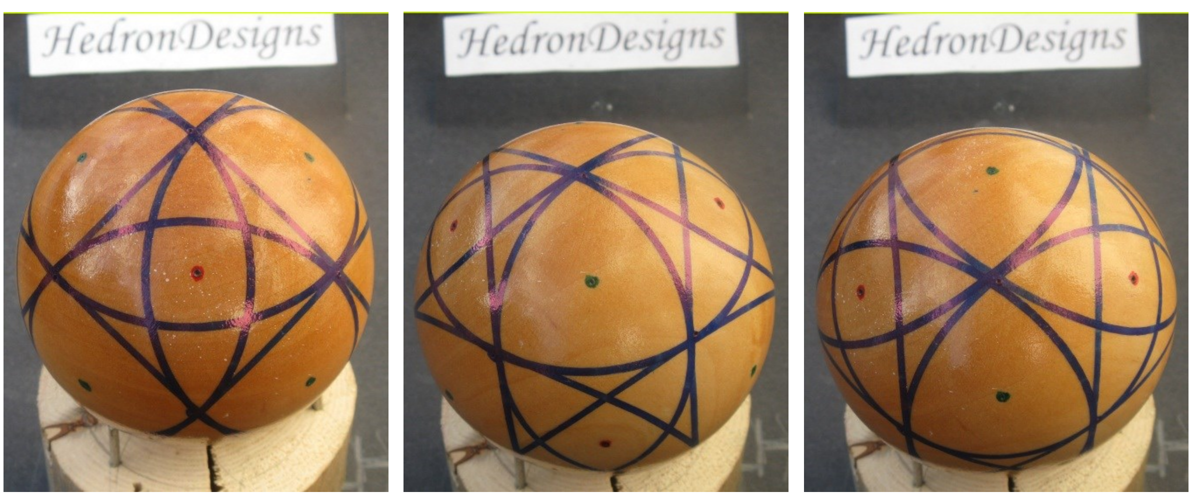
Vertex view **Face view** **Edge view**



$4E^{2f}$



$4E^{1v}$



4E

Vertex view **Face view** **Edge view**

$4E_{1v}$

Vertex view **Face view** **Edge view**

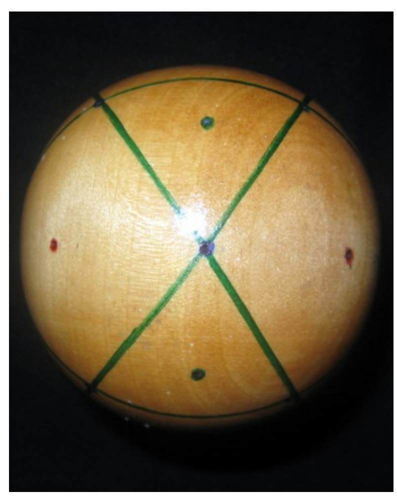


$3V^{1e}$



3V

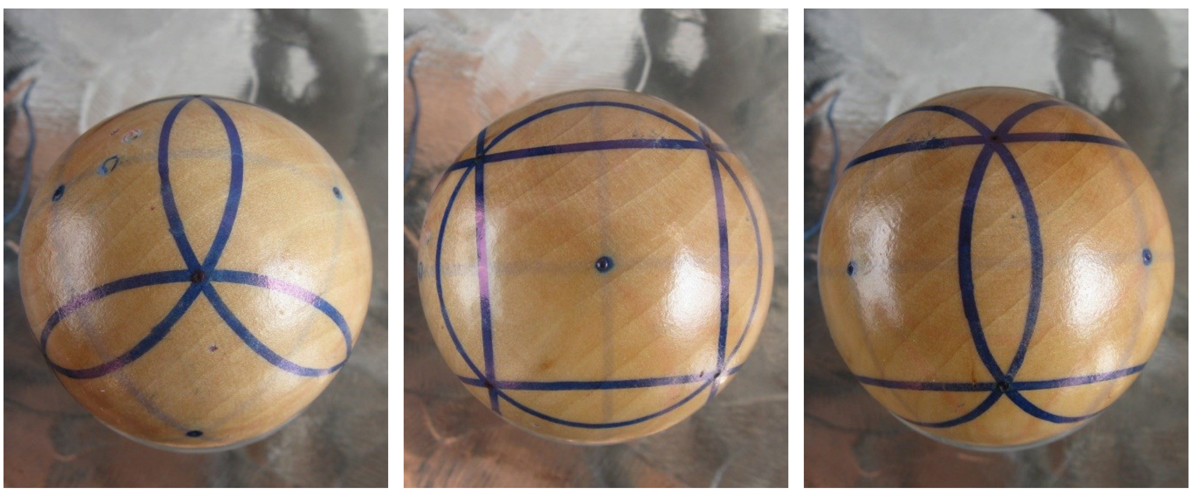




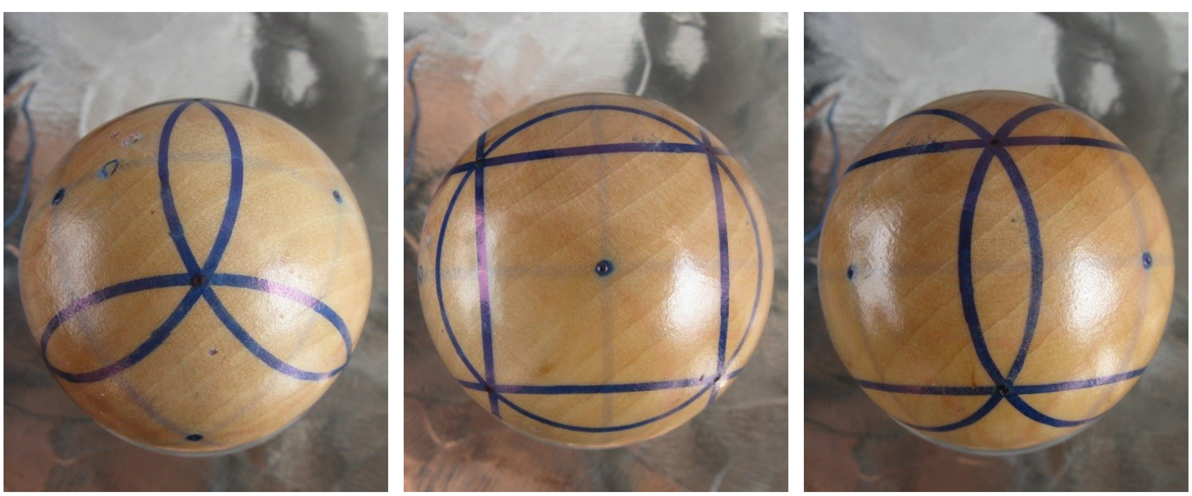
$3V_{1f}$

Vertex view **Face view** **Edge view**

$3E^{0e}$



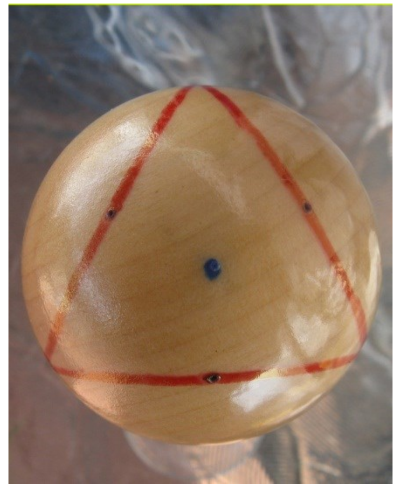
$3E^{1f}$



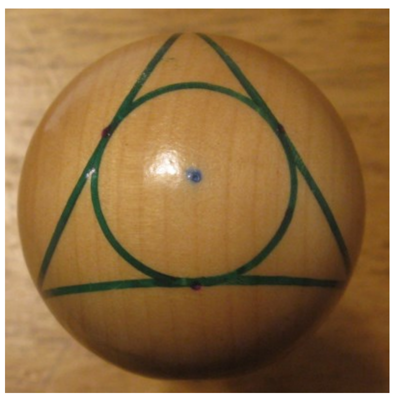
$3E^{1v}$

Vertex view **Face view** **Edge view**



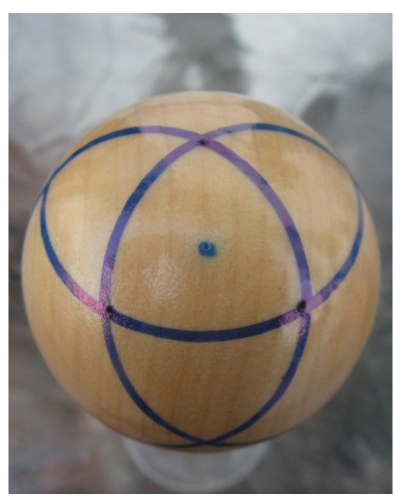


2V

2F*





2E*

Chapter 4: The Expansion Phases of the Polyspherons

In this chapter, I introduce a shape referred to as a polyspheron which consists of an arrangement of conix similar to harmonic notes. The polyspheron uses a different geometric reference for its construction. The unit conix will be the geometric reference and the sphere's radius R will vary in size. The harmonic notes seen in previous chapters use a unit sphere as its geometric reference and the size of the conix varied. There are eight different intervals of polyspherons (5V, 5F, 5E, 4V, 4F, 4E, 3V and 2V) that correspond to each of the intervals of harmonic notes.

When the conix in a polyspheron are concentric with the sphere, then the sphere's radius will be one unit and have a minimum volume. The polyspherons grow in size as the conix are re-arranged and their centers move away from the center of the sphere. Polyspherons can continue to grow as long as the conix touch each other and they reach their maximum volume when the conix abut each other. The nomenclature used for harmonic notes will be used for polyspherons as well.

The various polyspherons from each interval are listed in Table III. The first column contains the minimum volume polyspheron for each interval. The polyspherons increase in volume as you go left to right along each row.

Table III – Polyspheron Intervals

$2V_{2f}$	$2V_{1v}$	2V							
$3V_{1e}$	3V	$3V^{1e}$							
4V	$4V^{1f}$	$4V^{2e}$							
$4F_{1e}$	$4F^{0f}$	$4F^{1v}$	$4F^{2e}$						
$4E_{1vf}$	4E	$4E^{1v}$	$4E^{2f}$	$4E^{3}$					
$5V_{2e}$	$5V_{1f}$	5V	$5V^{1e}$	$5V^{2f}$	$5V^{3e}$				
$5F_{5e}$	$5F_{4v}$	$5F_{3f}$	$5F_{2v}$	$5F_{1e}$	5F	$5F^{1v}$	$5F^{2e}$		
$5E_{6e}$	$5E_{5e}$	$5E_{4f}$	$5E_{3e}$	$5E_{2v}$	$5E_{1f}$	5E	$5E^{1v}$	$5E^{2v}$	$5E^{3}$

Now I will describe the minimum volume polyspheron for each interval. The conix in each of these polyspherons is a "great circle" which divides the sphere into two equal hemispheres. The centers of the conix coincide with the center of the sphere and they both have a radius of one. There are no polyspherons smaller than these eight minimum volume polyspherons.

The simplest of the minimum volume polyspherons is $2V_{2f}$ which consists of three conix centered along the axes of the square faces of a regular triangular prism. Two groups of three conix intersect at the axes of the triangular faces of the prism. Refer to Figure 36 to see the triangular prism and two geometric models (trispheron and its harmonic note) of the $2V_{2f}$ polyspheron. Many other prism-based solids exist as well.

Figure 36

The next interval of polyspheron is 3V. The minimum volume polyspheron $3V_{1e}$ consists of four conix that have similar geometric properties as a circumscribed cuboctahedron, where groups of two circles intersect at its twelve vertices. Refer to Figure 37 to see the cuboctahedron and two geometric models (tetraspheron and its harmonic note) of the $3V_{1e}$ polyspheron.

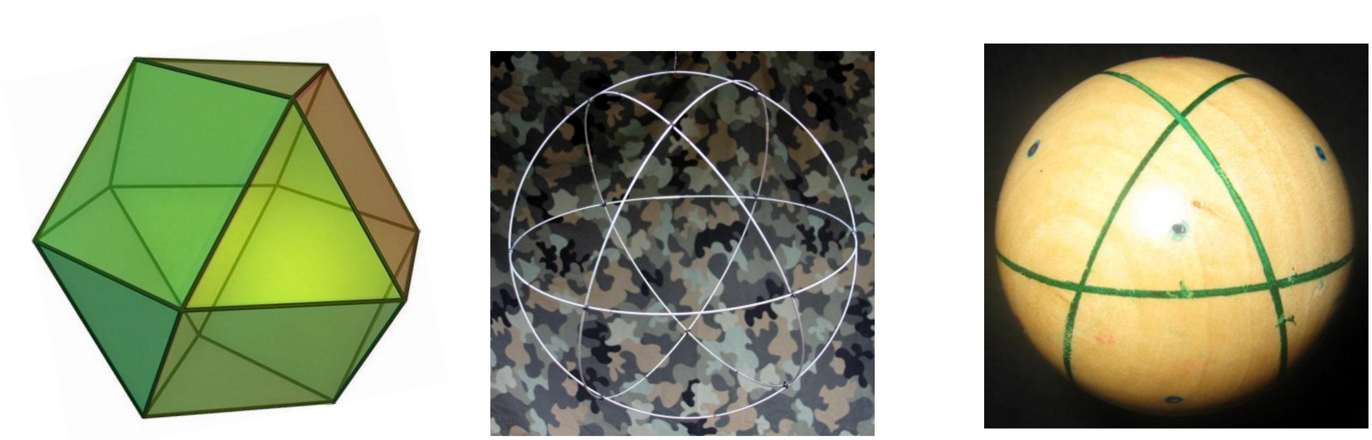


Figure 37

The next interval of polyspheron is 4V. The minimum volume polyspheron 4V consists of six conix that have similar geometric properties as a circumscribed octahedron, where groups of four circles intersect at its six vertices. Although there only appears to be three conix in this polyspheron, there are actually three pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 38 to see the related octahedron and two geometric models (hexaspheron and its harmonic note) of the 4V polyspheron.

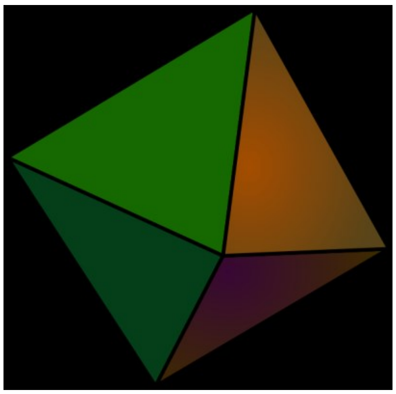
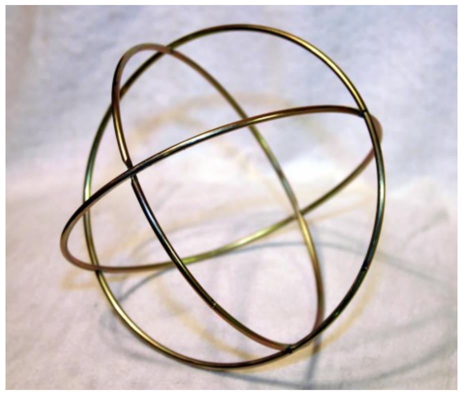
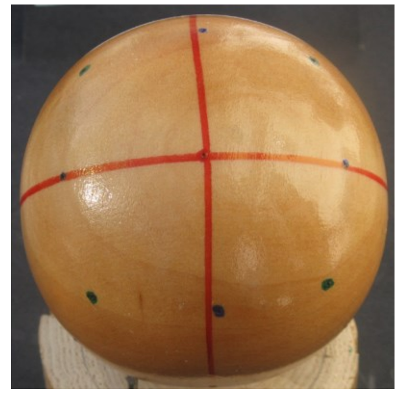




Figure 38

The next interval of polyspheron is 4F. The minimum volume polyspheron $4F_{1e}$ consists of eight conix that have similar geometric properties as a circumscribed cuboctahedron, where groups of four circles intersect at its twelve vertices. Although there only appears to be four conix in this polyspheron, there are actually four pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 39 to see the related cuboctahedron and two geometric models (octaspheron and its harmonic note) of the $4F_{1e}$ polyspheron.

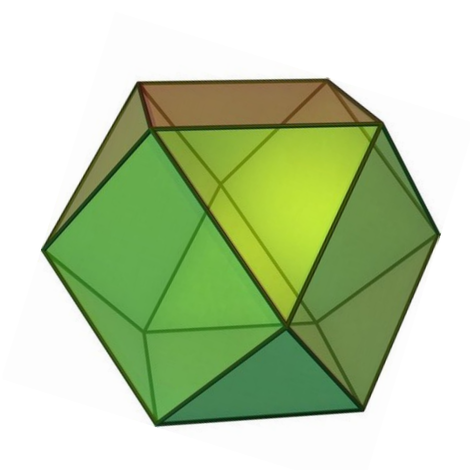
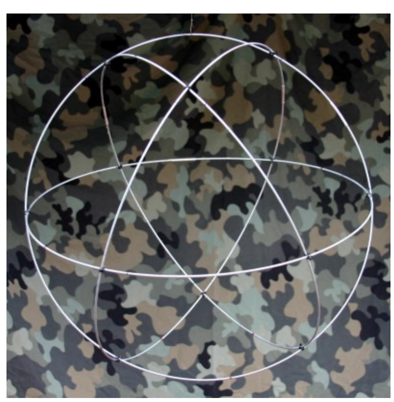
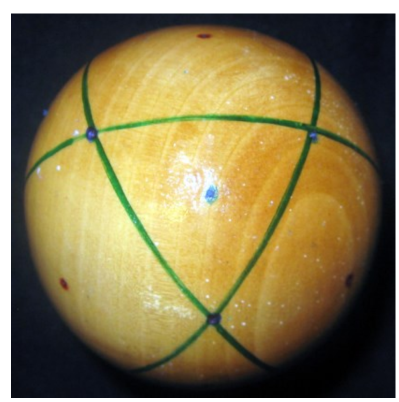




Figure 39

The next interval of polyspheron is 4E. The minimum volume polyspheron $4E_{1e}$ consists of twelve conix that have similar geometric properties as a circumscribed rhombic dodecahedron (dual of a cuboctahedron), where groups of two conix intersect at six vertices and groups of three conix intersect at the other eight vertices. Although there only appears to be six conix in this polyspheron, there are actually six pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 40 to see the related rhombic dodecahedron and one geometric model (its harmonic note) of the $4E_{1e}$ polyspheron.

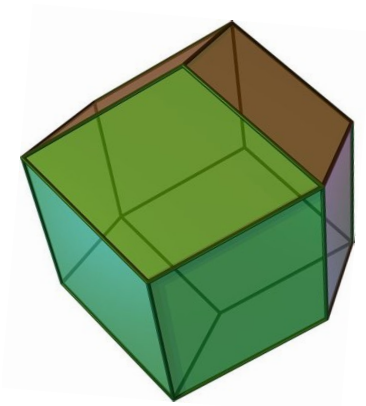
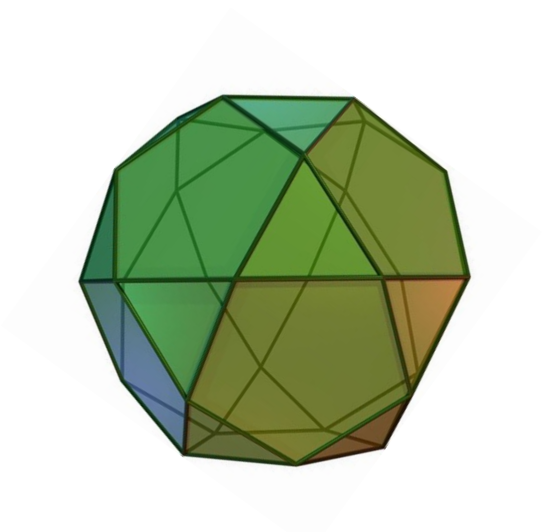


Figure 40

The next interval of polyspheron is 5V. The minimum volume polyspheron $5V_{2e}$ consists of twelve conix that have similar geometric properties as a circumscribed icosadodecahedron where groups of two conix intersect at its thirty vertices. Although there only appears to be six conix in this polyspheron, there are actually six pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 41 to see the related icosadodecahedron and two geometric models (dodecaspheron and its harmonic note) of the $5V_{2e}$ polyspheron.



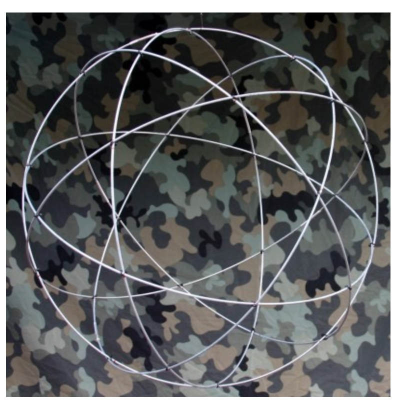


Figure 41

The next interval of polyspheron is 5F. The minimum volume polyspheron $5F_{5e}$ consists of twenty conix that have similar geometric properties as a circumscribed icosadodecahedron where groups of two conix intersect at its thirty vertices. Although there only appears to be ten conix in this polyspheron, there are actually ten pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 42 to see the related icosadodecahedron and one geometric model (its harmonic note) of the $5F_{5e}$ polyspheron.

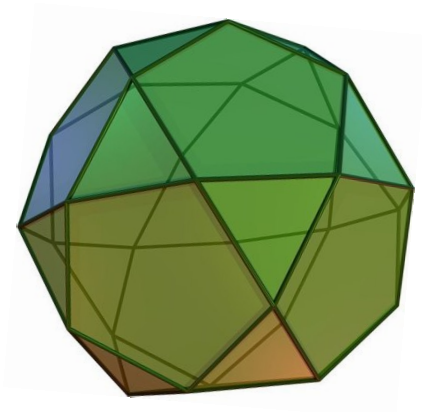
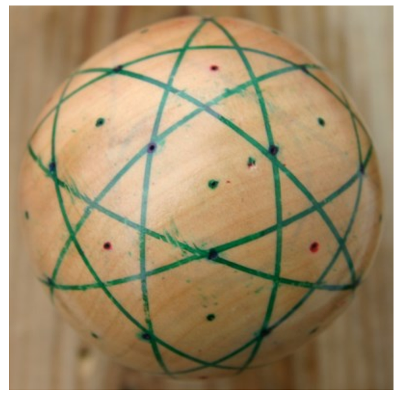



Figure 42

The next interval of polyspheron is 5E. The minimum volume polyspheron $5E_{6e}$ consists of thirty conix that have similar geometric properties as a circumscribed rhombic tricontahedron where groups of five conix intersect at its thirty vertices. Although there only appears to be fifteen conix in this polyspheron, there are actually fifteen pairs of overlapping circles that will become evident during the first expansion phase. Refer to Figure 43 to see the related dodecahedron and one geometric model (its harmonic note) of the $5E_{6e}$ polyspheron.

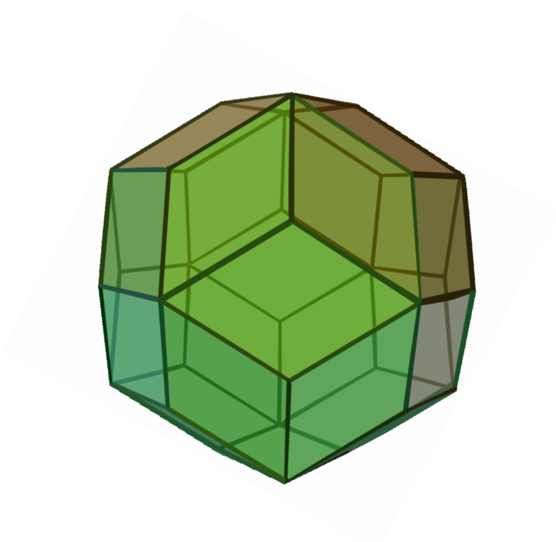
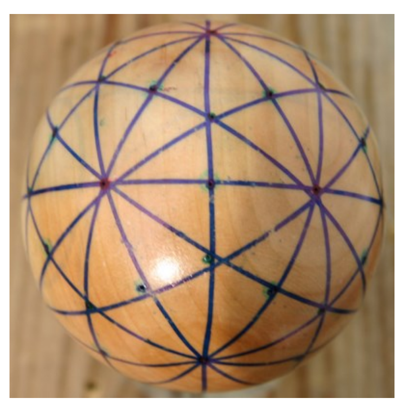



Figure 43

Now I will describe the first expansion phase of each of these polyspherons from their minimum volume to their maximum volume. During this expansion phase, the conix maintain their radius at one, but rearrange themselves on the surface of the polyspheron. This new arrangement reduces the amount of overlap between conix and as a result causes the polyspheron to grow in size as the center of its conix move away from the center of the sphere.

The 2V polyspheron

The 2V interval of polyspheron has three states ($2V_{2f}$ $2V_{1v}$ 2V) that are based on an arrangement of three unit conix. The minimum volume state is the $2V_{2f}$ polyspheron, where the sphere and conix centers share a single point. The conix are aligned on three different axes of a triangular prism and the conix and sphere has a radius of one unit. Refer to Figure 44 to see an illustration the three states ($2V_{2f}$ $2V_{1v}$ 2V).

The first growth step occurs as the conix start moving away from the center of the sphere. While the conix radius remains constant, the radius of the polyspheron grows from 1 to $\sqrt{7}/\sqrt{6}$. This is the $2V_{1v}$ polyspheron and it has similar geometric properties to a circumscribed triangular prism.

The second growth step occurs as the conix start moving further away from the center of the sphere. While the conix radius remains constant, the radius of the polyspheron grows from $\sqrt{7}/\sqrt{6}$ to $2/\sqrt{3}$. This is the 2V polyspheron and it has similar geometric properties to a midradius triangular prism and is the maximum volume state of the 2V interval of polyspheron. In its maximum state and polyspheron's conix have no overlap, but they abut fully.

Figure 44

I have put together a composite view of all three states of the 2V polyspheron, super imposed in the illustration in Figure 45, where the individual expansion steps can be seen relative to each other. The $2V_{2f}$ polyspheron is located at the center of illustration. The white arrows highlight the first expansion step ($2V_{2f}$ to $2V_{1v}$) as the conix move away from the center of the polyspheron. The red arrows highlight the second expansion step ($2V_{1v}$ to 2V) as the conix move even further from the center of the polyspheron. During the first expansion phase, the polyspheron's expansion is halted when the conix overlap is eliminated and the conix become fully abutted.

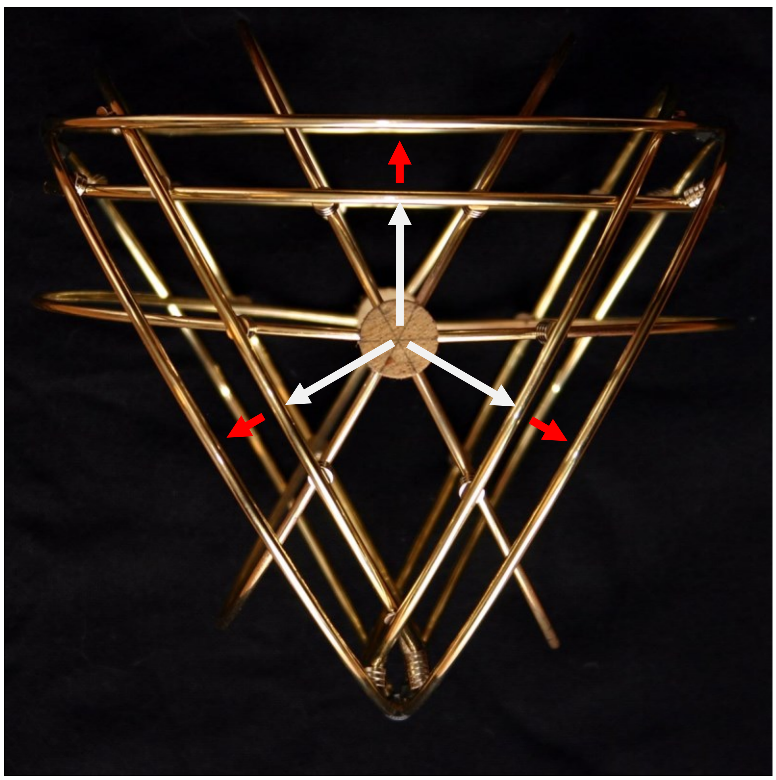


Figure 45

The other seven polyspherons grow in a similar fashion and in various growth steps. Table IV lists the various polyspheron for each interval, the polyspheron's radius and the polyspheron's reference shape. The reference shape type could be inscribed, midradius or circumscribed.

Polyspheron	Shape	Type	Radius
$2V_{2f}$	Triangular prism	Inscribed	1
$2V_{1v}$	Triangular prism	Circumscribed	$\sqrt{7}/\sqrt{6}$
2V	Triangular prism	Midradius	$2/\sqrt{3}$
$3V_{1f}$	Cuboctahedron	Circumscribed	1
3V	Tetrahedron	Circumscribed	$3/2\sqrt{2}$
$3V^{1e}$	Tetrahedron	Midradius	$\sqrt{3}/\sqrt{2}$

Table IV

Polyspheron	Shape	Type	Radius
4V	Octahedron	Circumscribed	1
$4V^{1f}$	Hexahedron	Circumscribed	$\sqrt{3}/2$
$4V^{2e}$	Hexahedron	Midradius	$\sqrt{2}$
$4F_{1e}$	Cuboctahedron	Circumscribed	1
4F	Stellated Octahedron	Circumscribed	$3/2\sqrt{2}$
$4F^{1v}$	Octahedron	Circumscribed	$\sqrt{3}/\sqrt{2}$
$4F^{2e}$	Octahedron	Midradius	$\sqrt{3}$
$4E_{1e}$	Rhombic Dodecahedron	Circumscribed	1
4E	Cuboctahedron	Circumscribed	$2/\sqrt{3}$
$4E^{1f}$	Octahedron	Circumscribed	$\sqrt{2}$
$4E^{2v}$	Octahedron	Inscribed	$\sqrt{3}$
$4E^{3}$	Rhombicuboctahedron	Circumscribed	2

Table IV cont'd

Polyspheron	Shape	Type	Radius
$5V_{2e}$	Icosadodecahedron	Circumscribed	1
$5V_{1f}$	Icosahedron	Midradius	$(\sqrt{15+6\sqrt{5}})/(\sqrt{14+6\sqrt{5}})$
5V	Icosahedron	Circumscribed	$\sqrt{5}/2$
$5V^{1e}$	Icosahedron	Inscribed	$(\sqrt{10+2\sqrt{5}})/(1+\sqrt{5})$
$5V^{2f}$	Dodecahedron	Circumscribed	$(\sqrt{15-3\sqrt{5}})/(\sqrt{5}-1)(\sqrt{2})$
$5V^{3e}$	Dodecahedron	Midradius	$(\sqrt{10-2\sqrt{5}})/(\sqrt{5}-1)$
$5F_{5e}$	Icosadodecahedron	Circumscribed	1
$5F_{4v}$	Snub Dodecahedron	Inscribed	$(\sqrt{15+6\sqrt{5}})/(\sqrt{14+6\sqrt{5}})$
$5F_{3f}$	Dodecahedron	Circumscribed	$3/(2\sqrt{2})$
$5F_{2e}$	Dodecahedron	Circumscribed	$(\sqrt{6}(\sqrt{3}+\sqrt{5}))/(3+\sqrt{5})$
$5F_{1e}$			$\sqrt{3}/\sqrt{2}$
5F	Dodecahedron	Inscribed	$3/2$
$5F^{1v}$	Icosahedron	Circumscribed	$(\sqrt{15-3\sqrt{5}})/(\sqrt{5}-1)(\sqrt{2})$
$5F^{2e}$	Icosahedron	Midradius	$(2\sqrt{3})/(\sqrt{5}-1)$
$5E_{6e}$	Dodecahedron	Inscribed	1
$5E_{5e}$	Rhombicuboctahedron		$1/\sin(2\pi/5)$
$5E_{4f}$	Rhombicuboctahedron		$(\sqrt{6}(\sqrt{3}+\sqrt{5}))/(3+\sqrt{5})$
$5E_{3e}$	Icosadodecahedron		$2/\sqrt{3}$
$5E_{2v}$	Icosadodecahedron		$(\sqrt{10+2\sqrt{5}})/(1+\sqrt{5})$
$5E_{1f}$	Icosadodecahedron		$\sqrt{3}/\sqrt{2}$
5E	Icosadodecahedron	Inscribed	$1/\sin(\pi/5)$
$5E^{1v}$	Dodecahedron	Inscribed	$(\sqrt{10-2\sqrt{5}})/(\sqrt{5}-1)$
$5E^{2f}$	Rhombicosadodecahedron	Inscribed	$(2\sqrt{3})/(\sqrt{5}-1)$
$5E^{3}$	Rhombicosadodecahedron	Circumscribed	$1+\sqrt{5}$

Table IV cont'd

The 3V polyspheron

Refer to Figure 46 for the various volume states of the 3V polyspheron. I scaled the sizes of its harmonic notes to represent the growth in size of these polyspherons.

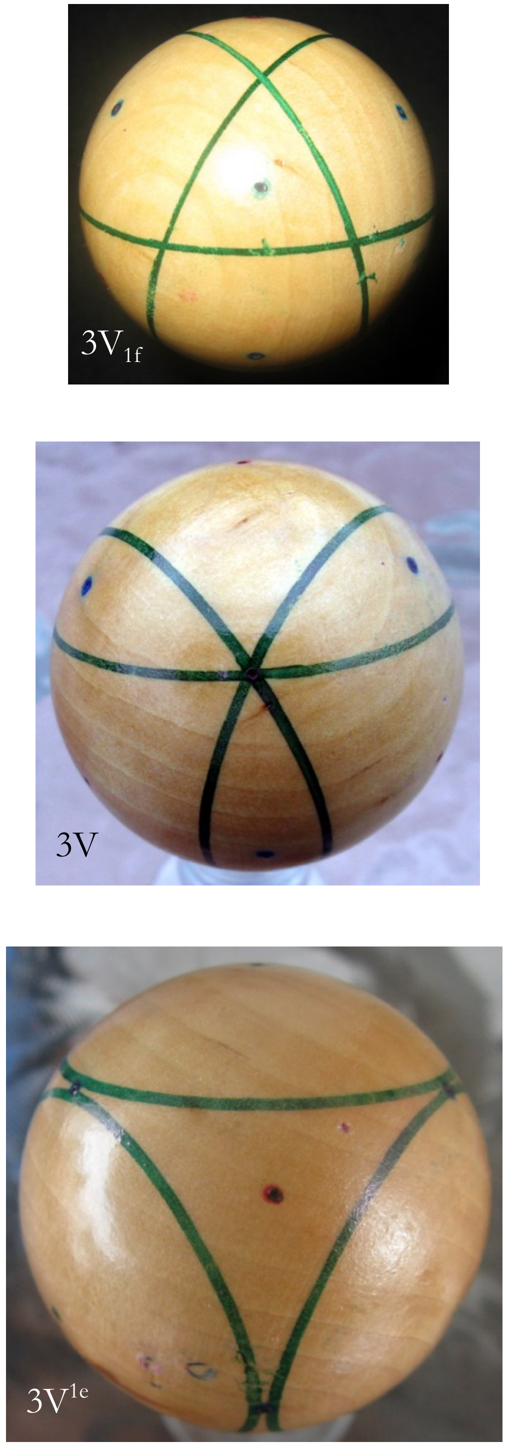


Figure 46

I have put together a composite view of all three states of the 3V polyspheron, super imposed in the illustration in Figure 47, where the individual expansion steps can be seen relative to each other. The white arrows highlight the first expansion step ($3V_{1e}$ to 3V) as the conix move away from the center of the polyspheron. The red arrows highlight the second expansion step (3V to $3V^{1e}$) as the conix move even further from the center of the polyspheron.

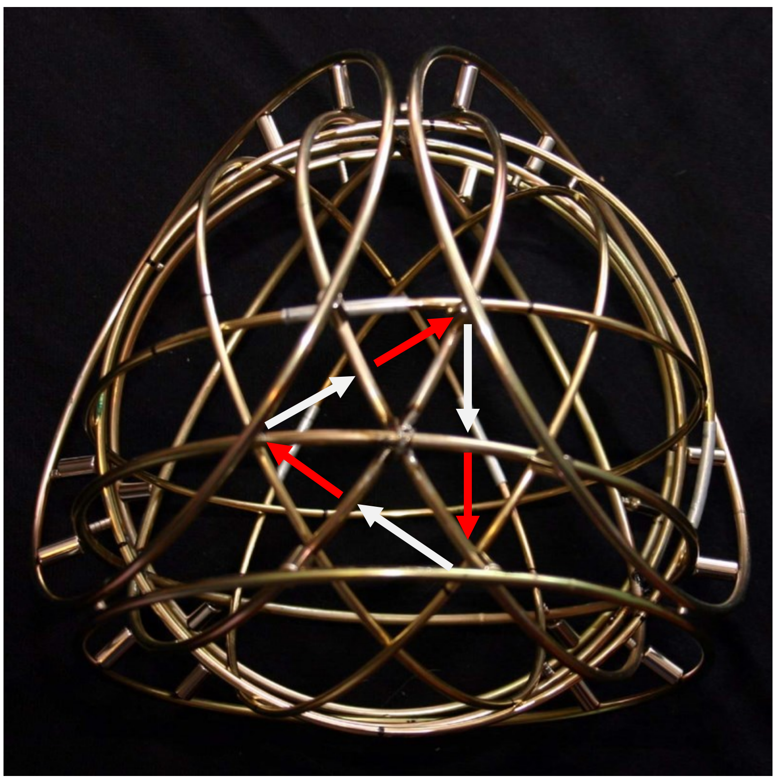
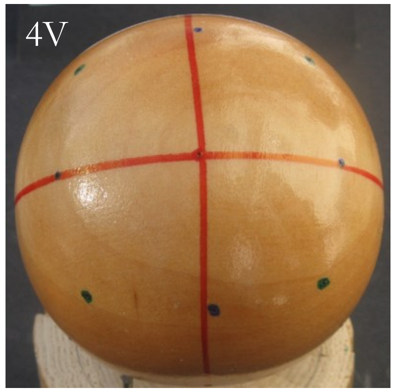
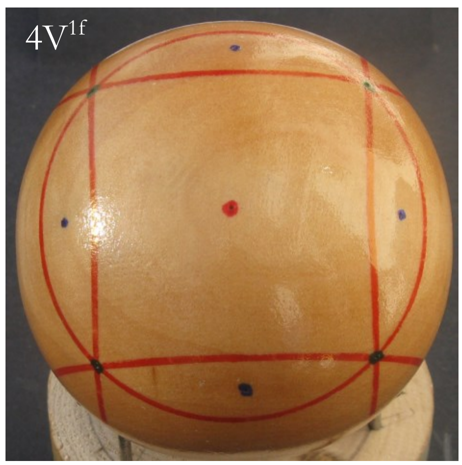


Figure 47

The 4V polyspheron

Refer to Figure 48 for the various volume states of the 4V polyspheron. I scaled the sizes of 4th harmonic notes to represent the growth in size of these polyspherons.





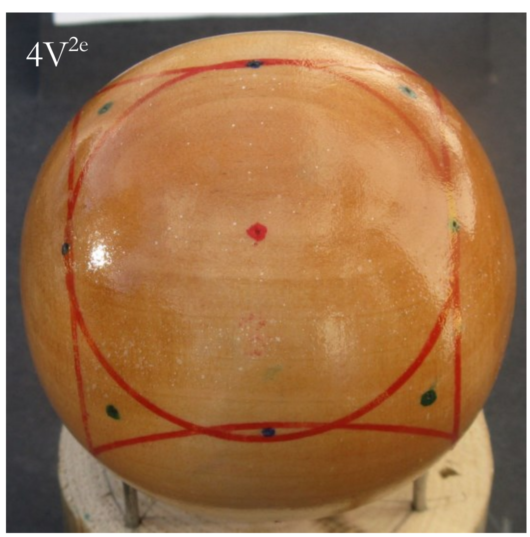


Figure 48

I have put together a composite view of all three volume states of the 4V polyspheron, super imposed in the illustration in Figure 49, where the individual expansion steps can be seen relative to each other. The $4V^{0v}$ polyspheron is located at the center of image. The white arrows highlight the first expansion step (4V to $4V^{1f}$) as the conix move away from the center of the polyspheron. Notice how what appeared to be a single conix split in two and moves in different directions. The red arrows highlight the second expansion step ($4V^{1f}$ to $4V^{2e}$) as the conix move even further from the center of the polyspheron.

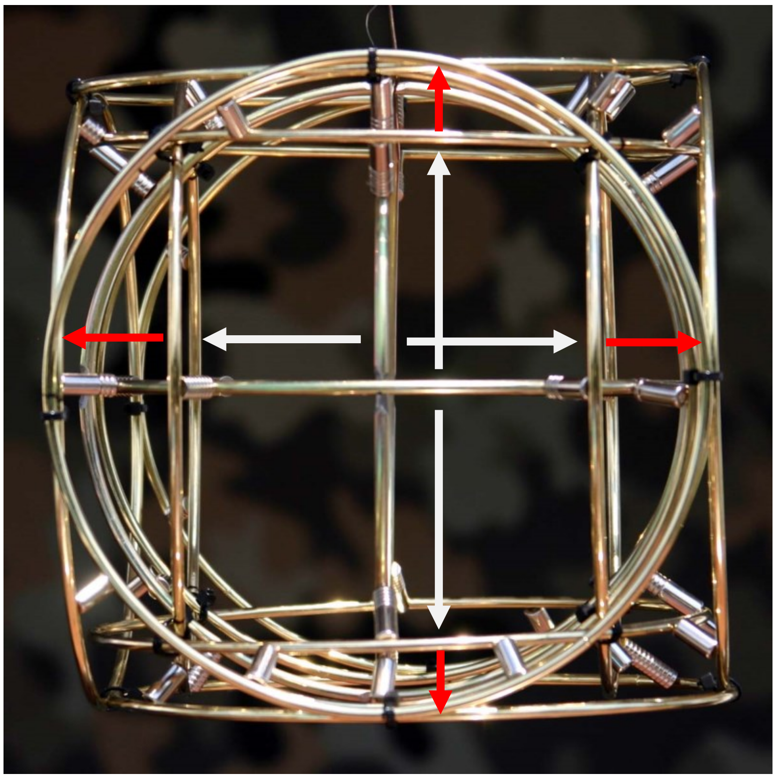


Figure 49

The 4F polyspheron

Refer to Figure 50 for the various volume states of the 4F polyspheron. I scaled the sizes of its 4th harmonic notes to represent the growth in size of these polyspherons.

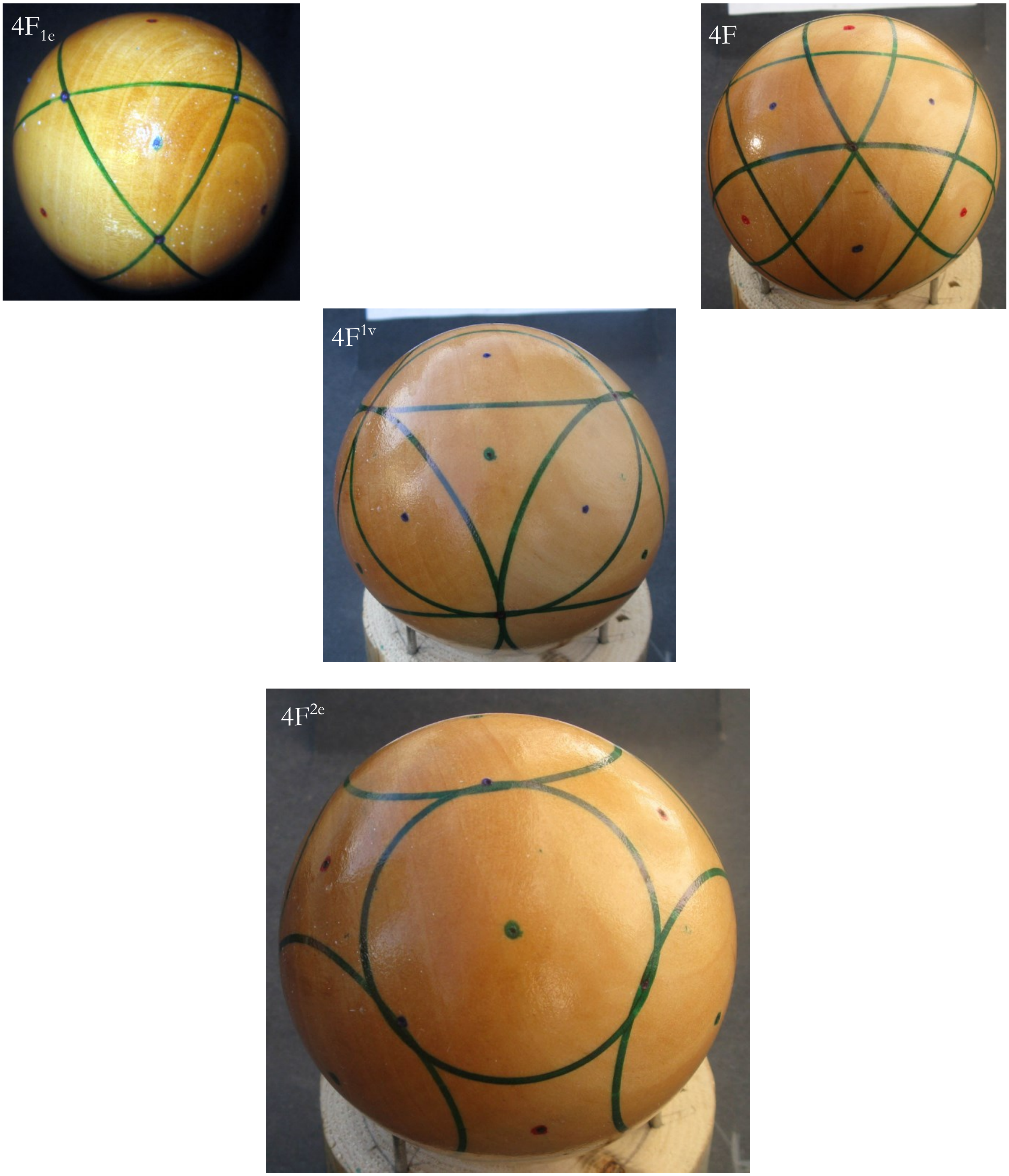


Figure 50

I have put together a composite view of the $4F_{1e}$ and $4F^{2e}$ polyspherons refer to Figure 51-52.

Figure 51

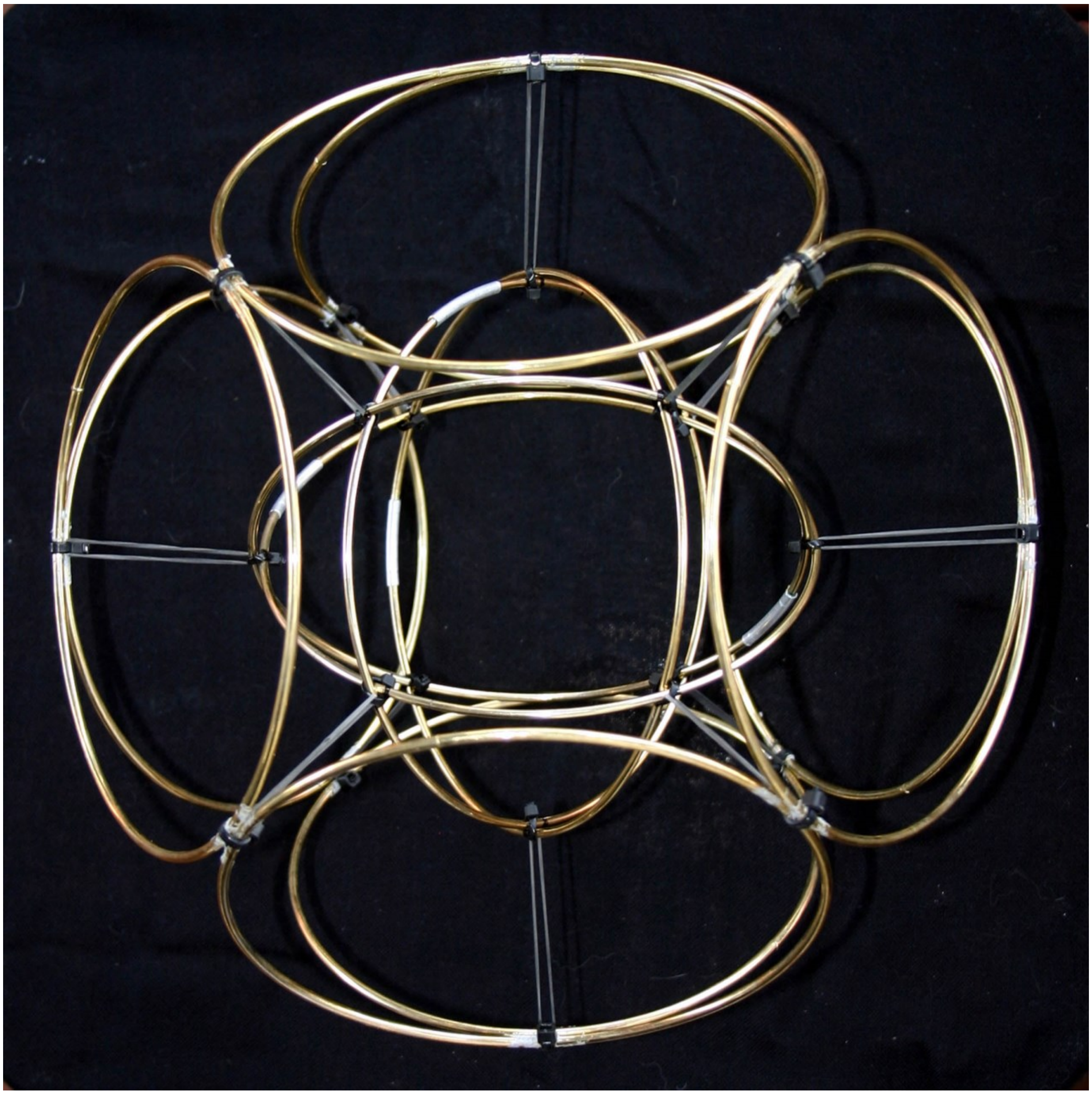
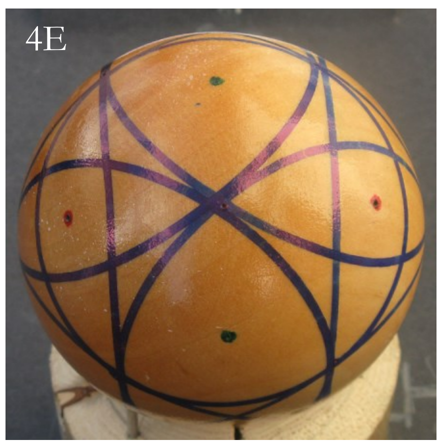


Figure 52

The 4E polyspheron

Refer to Figure 53-54 for the various volume states of the 4E polyspheron. I scaled the sizes of its 4th harmonic notes to represent the growth in size of these poly-spherons.



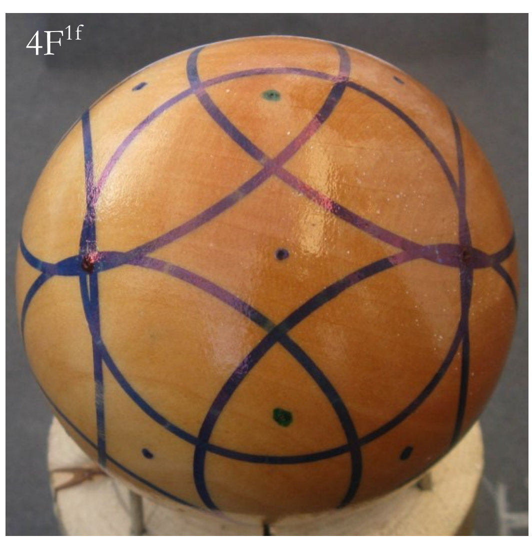


Figure 53

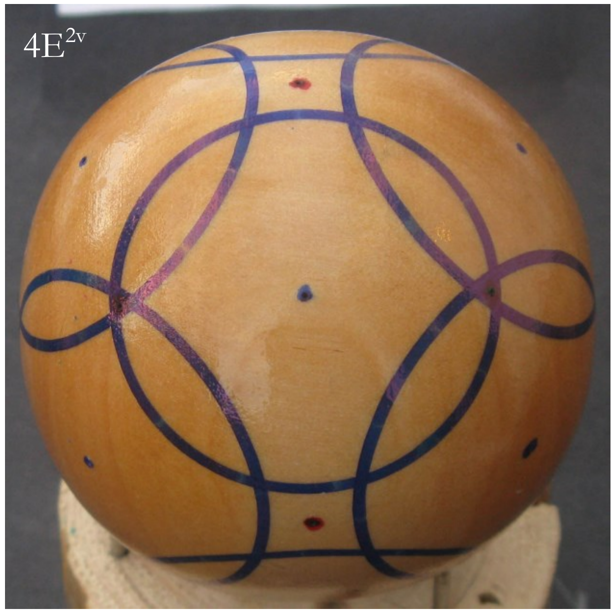


Figure 54

5^{th} Harmonic V-Note: The 5V polyspheron

Refer to Figure 55 for the various volume states of the 5V polyspheron. I scaled the sizes of its 5^{th} harmonic notes to represent the growth in size of these polyspherons.

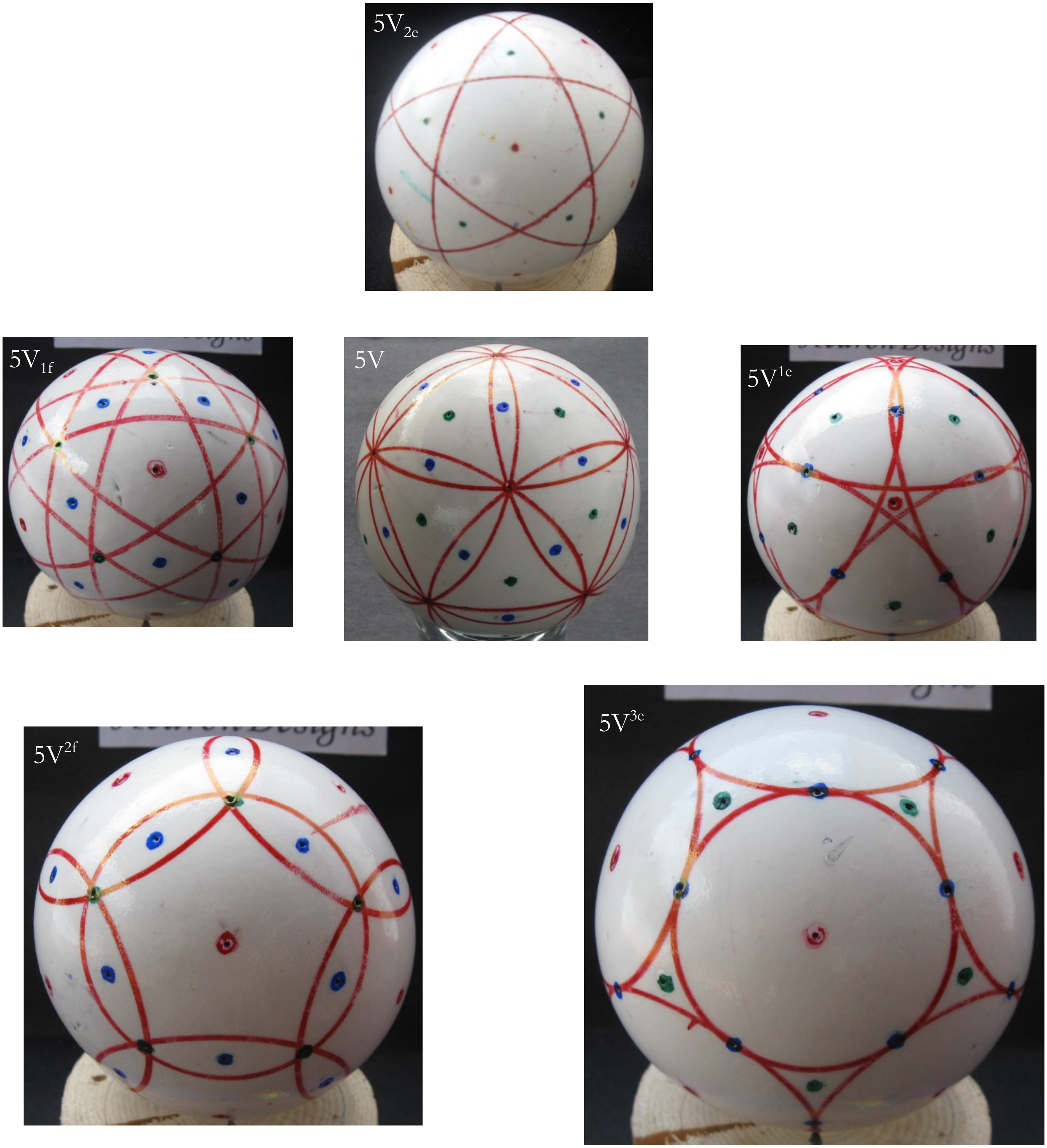


Figure 55

Refer to Figure 56-58 for various views of the $5V^{3e}$ polyspheron using a dodecaspheron model consisting of twelve metal rings.

Figure 56

Figure 57

Figure 58

I have put together a composite view of the $5V_{2e}$ and $5V^{3e}$ polyspherons refer to Figure 59-61.

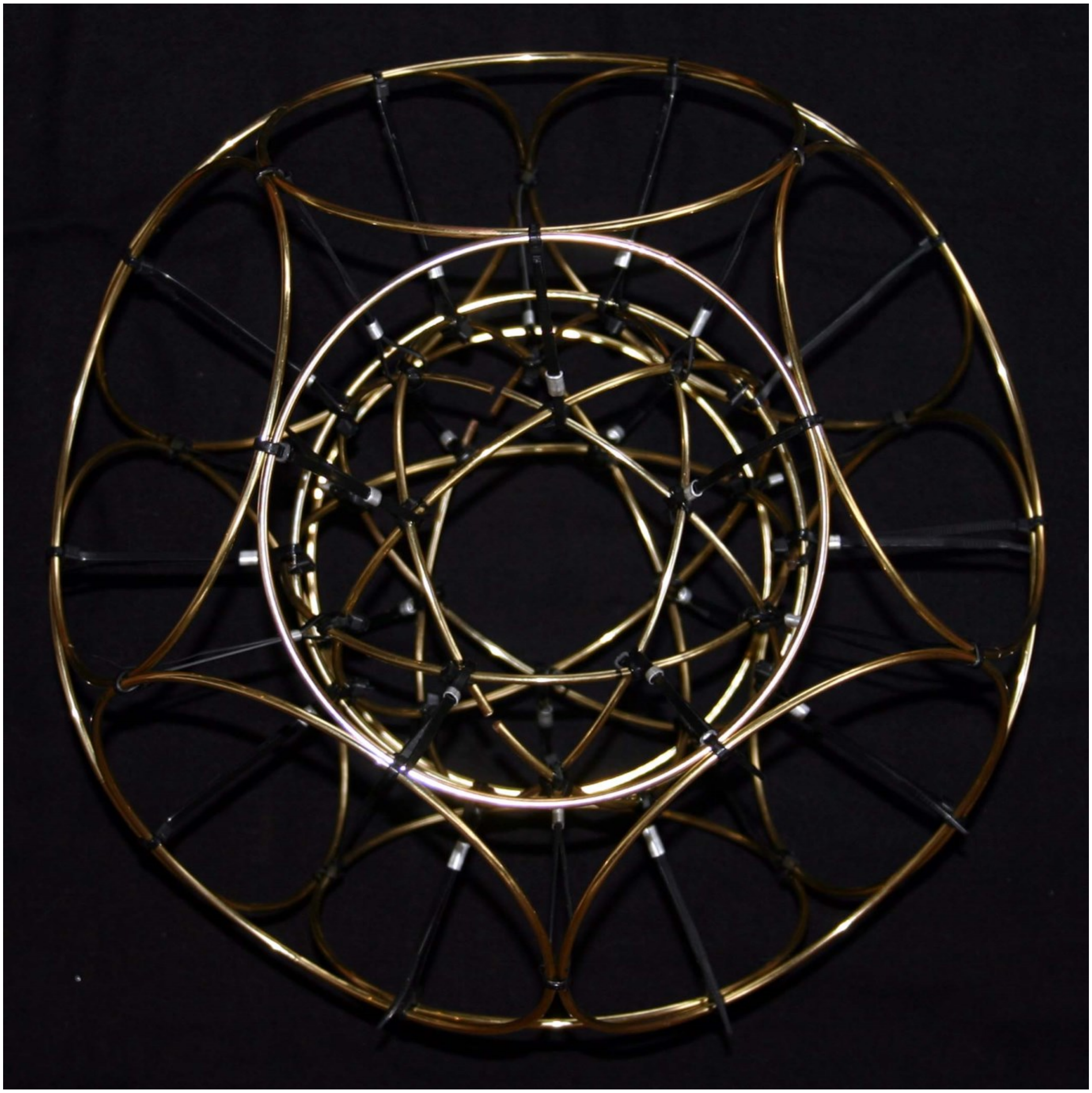


Figure 59

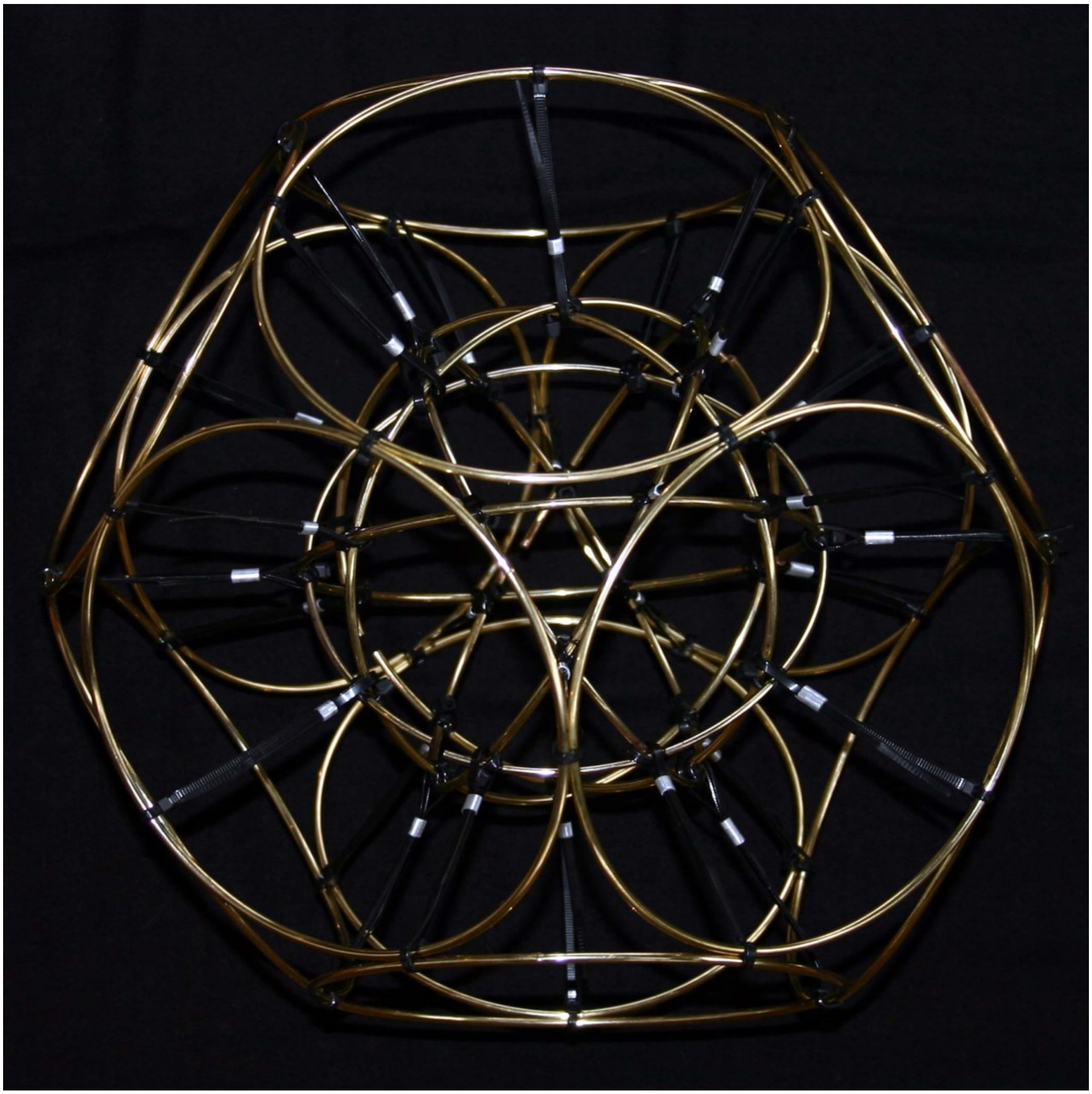


Figure 60

Figure 61

The 5F polyspheron

Refer to Figure 62-63 for the various volume states of the 5F polyspheron. I scaled the sizes of its 5th harmonic notes to represent the growth in size of these polyspherons.

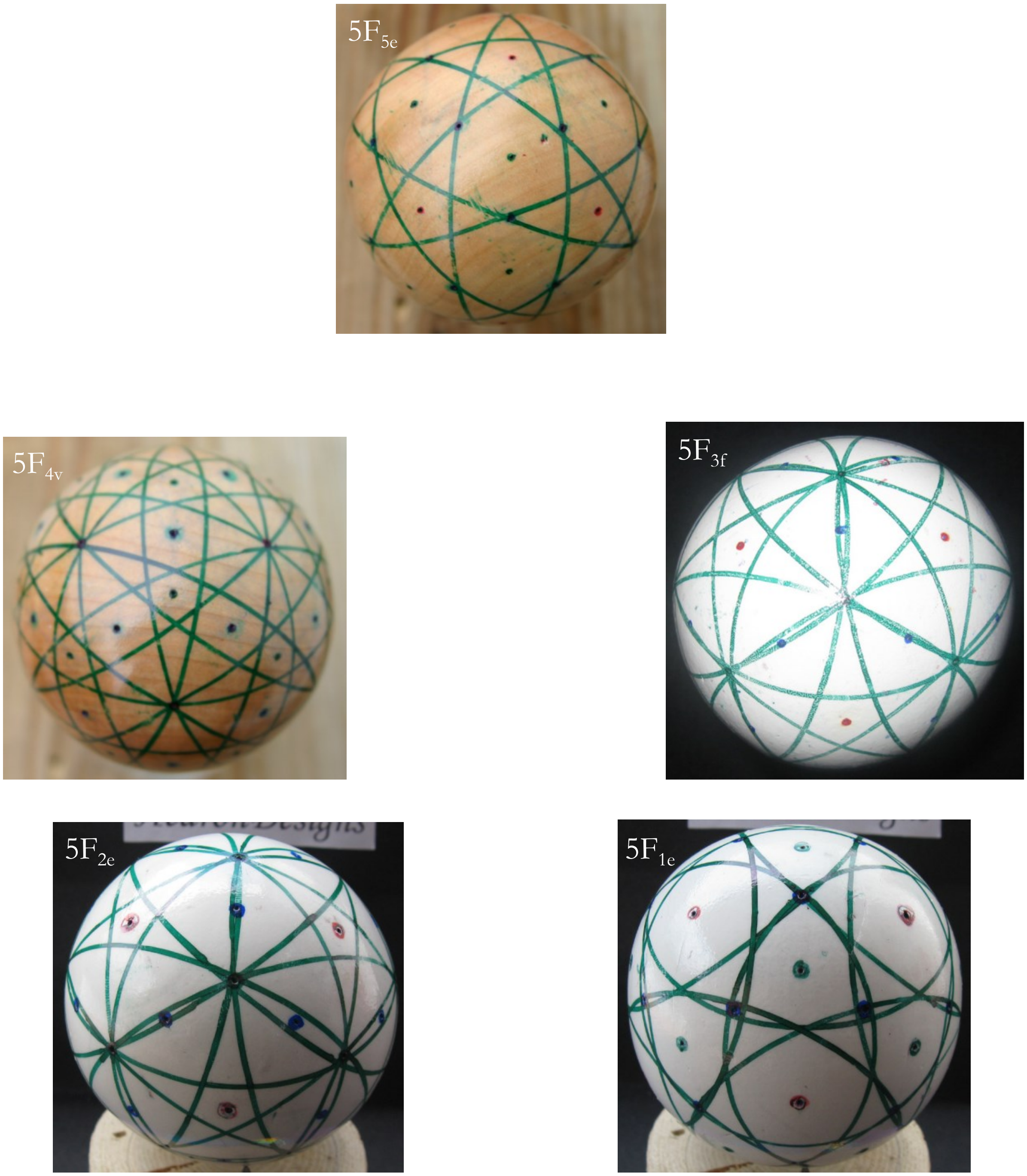
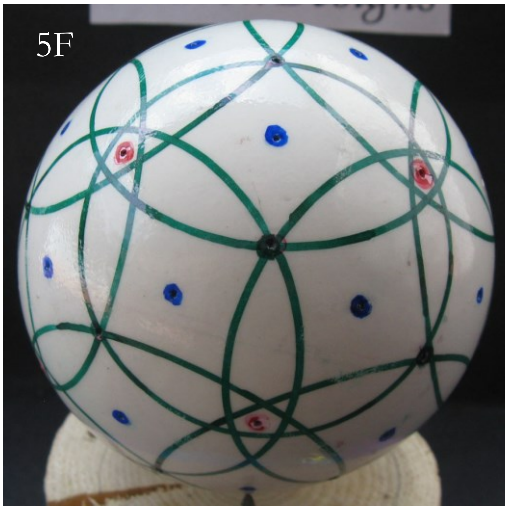


Figure 62



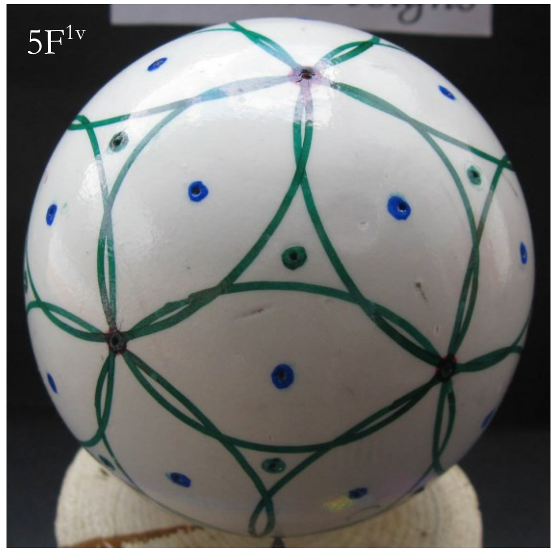


Figure 63

5^{th} Harmonic E-Note: The 5E polyspheron

Refer to Figure 64-66 for the various volume states of the 5E polyspheron. I scaled the sizes of its 5^{th} harmonic notes to represent the growth in size of these poly-spherons.

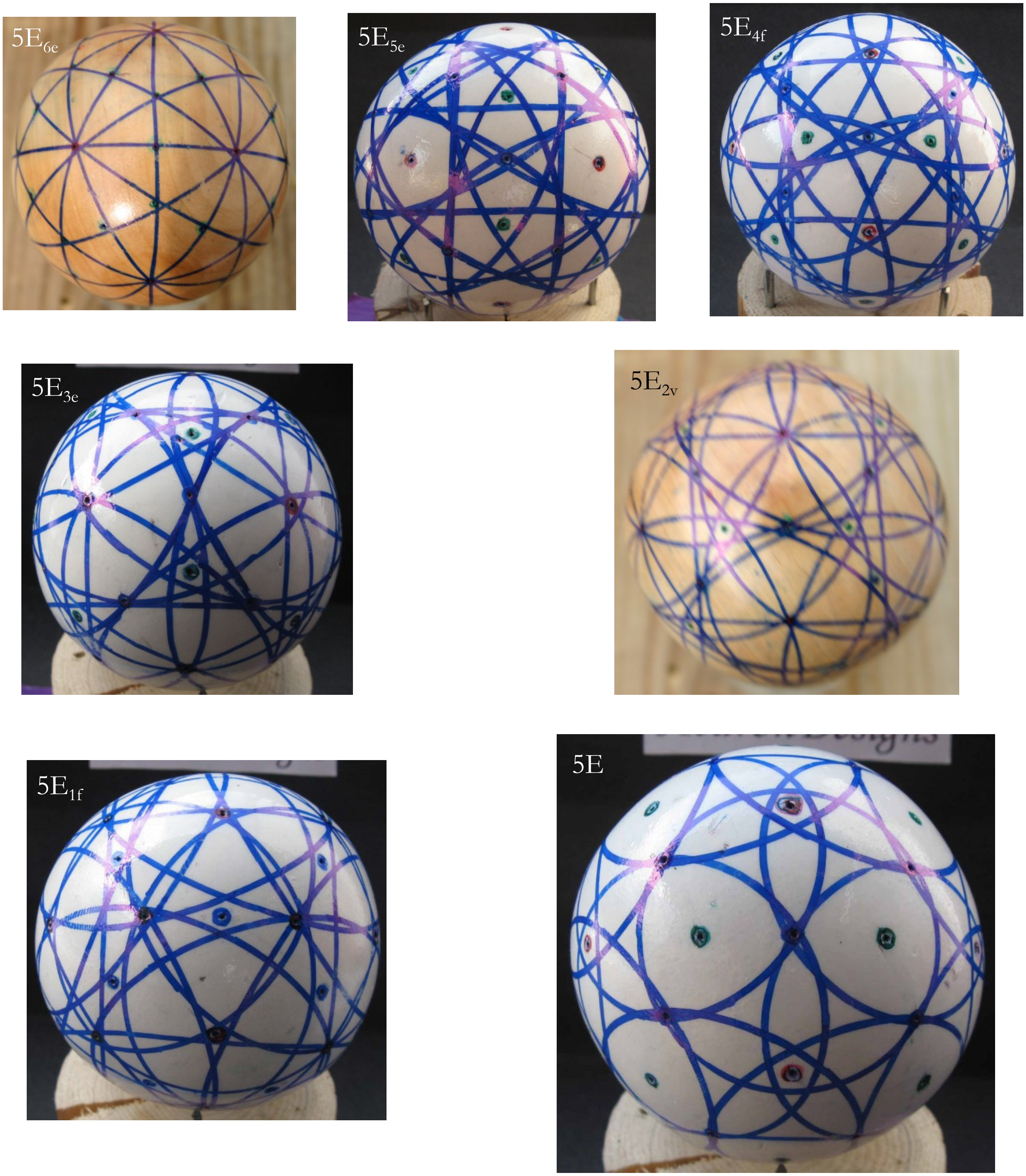


Figure 64

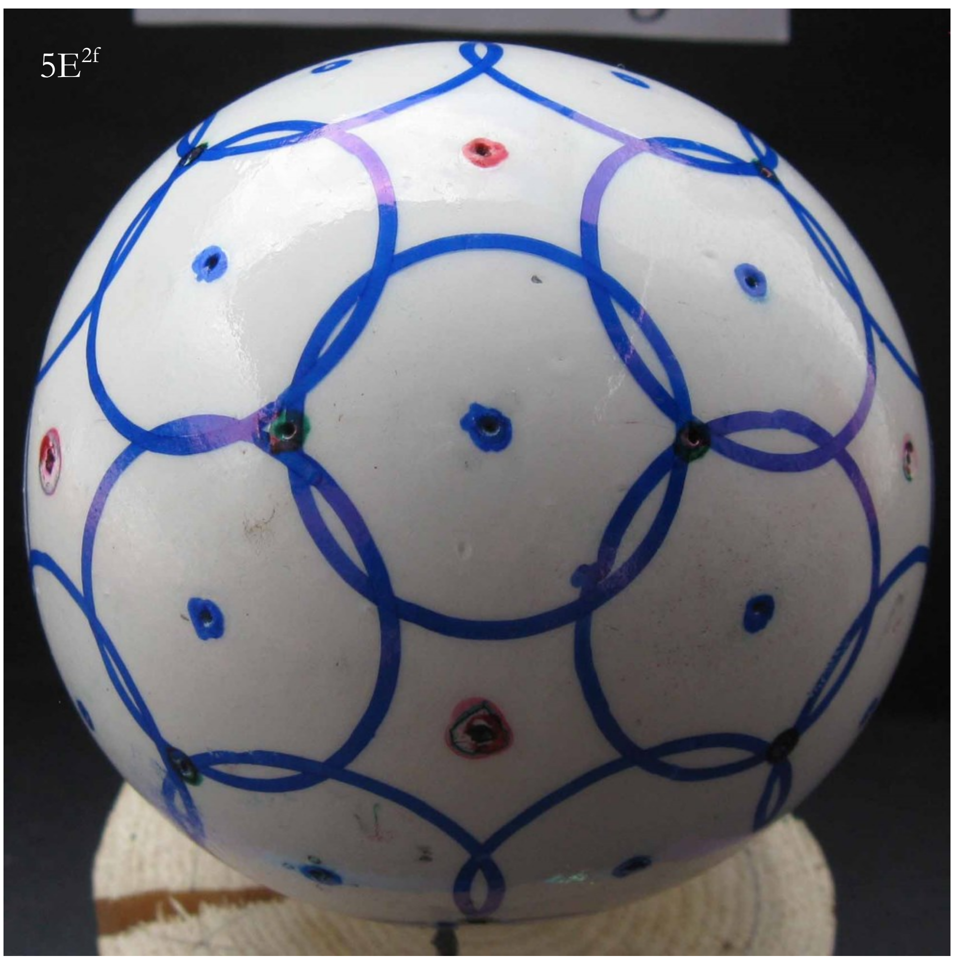


Figure 65

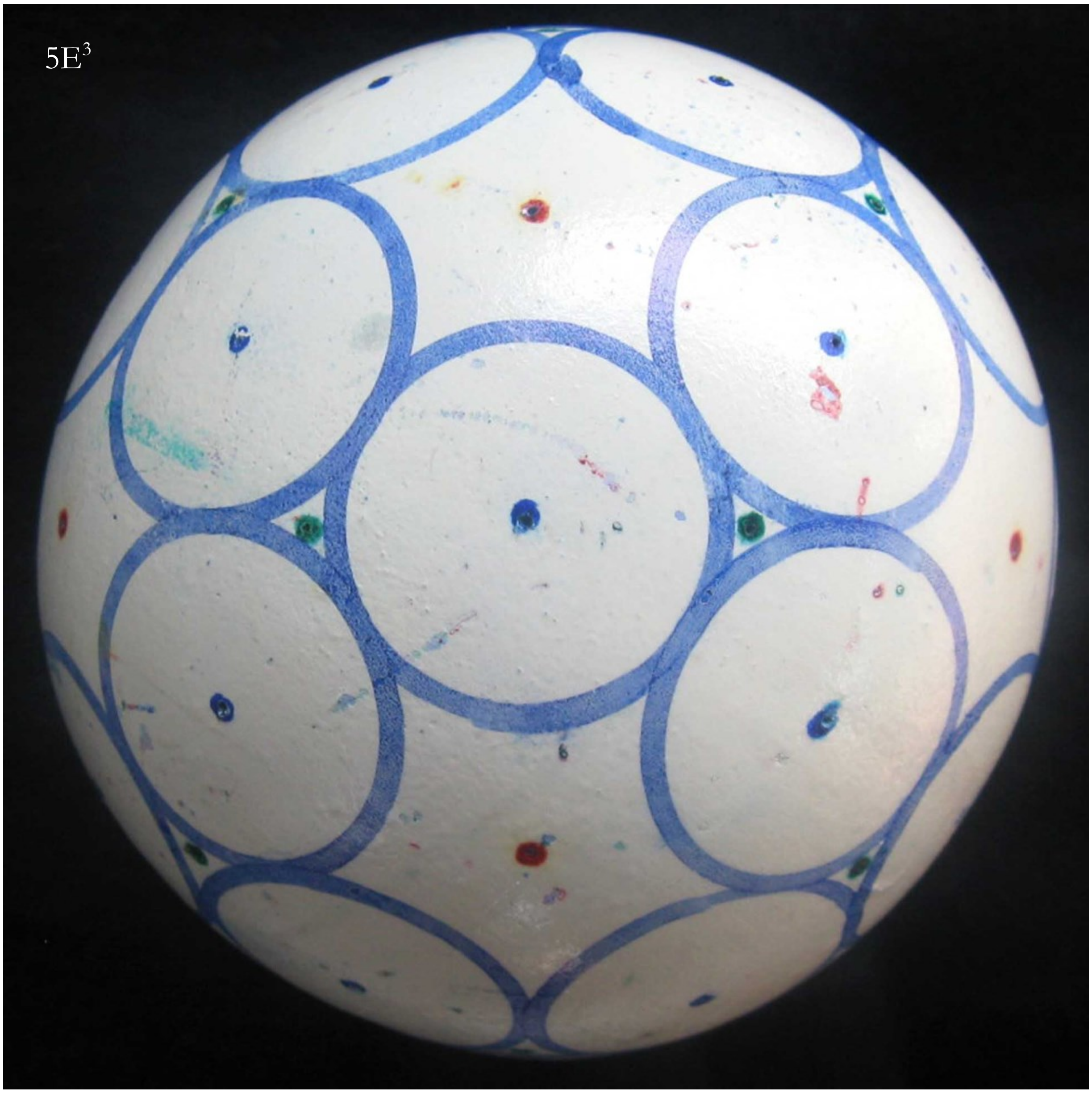


Figure 66

Chapter 5: The Transformation of Polyspheron to Polyconix

When I look at each maximum volume polyspheron, it suggests to me that they could be comprised of cones. I would also suggest that these cones are created as the conix in these polyspherons grow at a uniform rate.

As the conix grow in size, the polyspherons grow as well, and at the same time the conix move away from the center of the polyspheron which traces out cone shapes that radiate from its center. Refer to Figure 67 to see an example of how four cones would be created by the expanding conix and its growing polyspheron. One cone is created by each conix and these polyspherons transform them into what will be referred to as a polyconix. There is one unique polyconix for each of the eight maximum volume polyspherons.

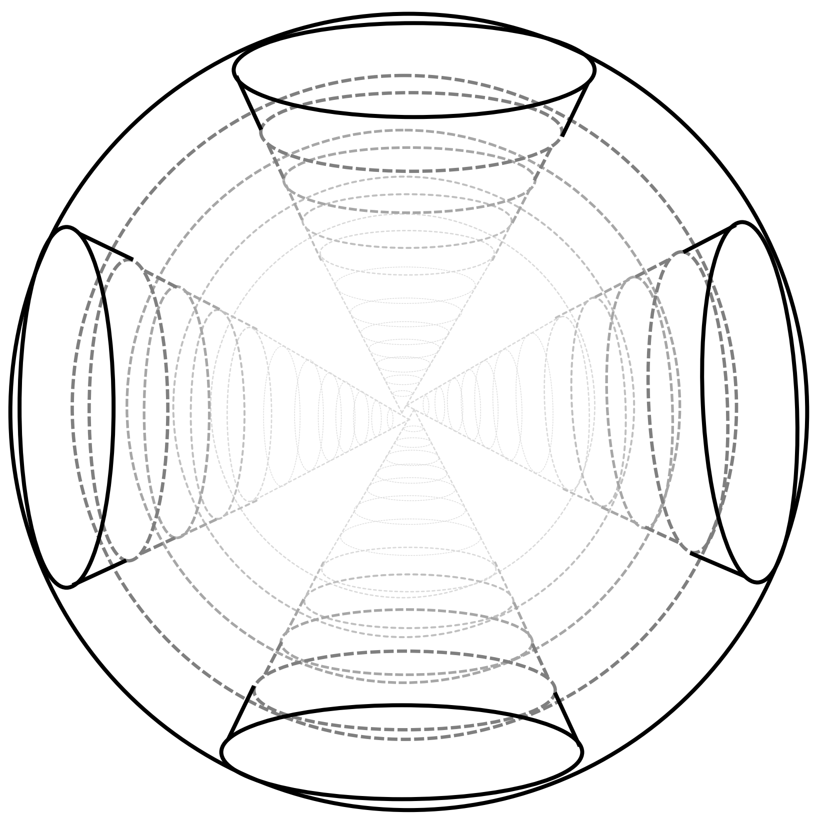


Figure 67

The eight polyconix are listed in Table V. The first column lists the polyspheron at its maximum volume which is the basis for each polyconix. The second column lists the name of each specific polyconix and the third column lists the polyhedrons with which it shares common geometric properties.

Polyspheron	**Polyconix Name**	**Reference Polyhedron**
$2V^{0f}$	triconix	triangular prism
$3V^{1e}$	tetraconix	tetrahedron
$4V^{2e}$	hexaconix	hexahedron
$4F^{2e}$	octaconix	octahedron
$4E^{3}$	predodecaconix	rhombicuboctahedron
$5V^{3e}$	dodecaconix	dodecahedron
$5F^{2e}$	icosaconix	icosahedron
$5E^{3}$	tricontaconix	rhombicosadodecahedron

Table V

In Figure 68-75 you will see how each polyspheron transforms itself into a polyconix.

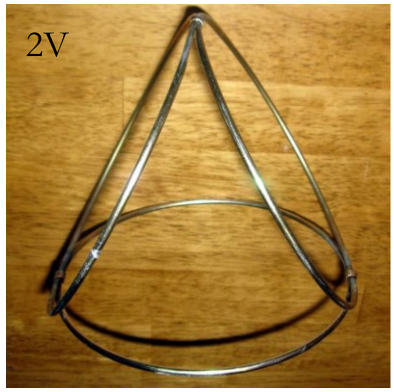


Figure 68

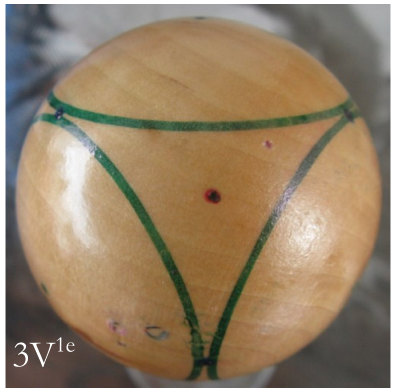
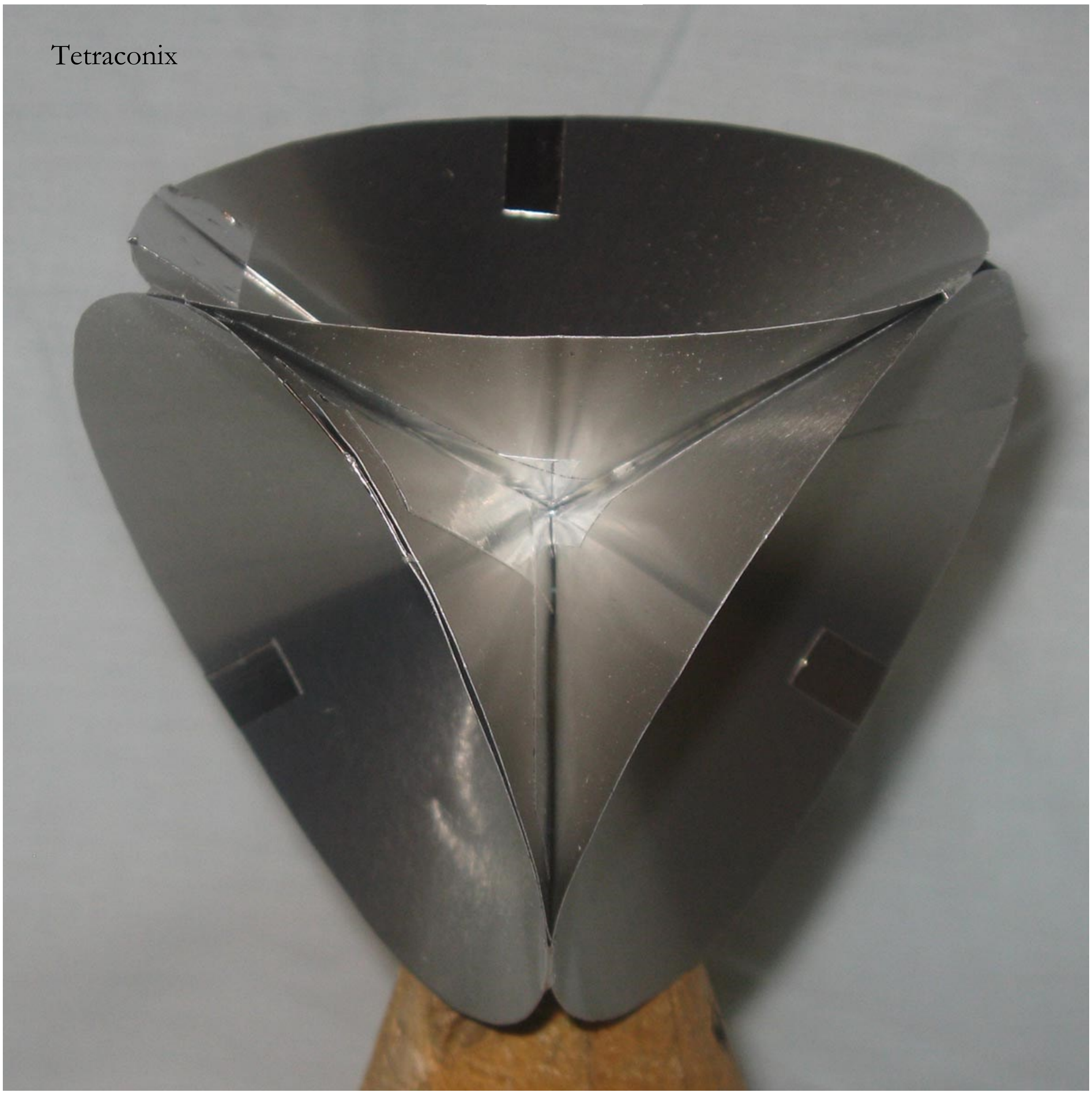


Figure 69



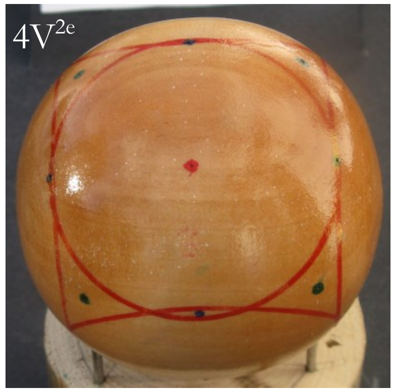


Figure 70

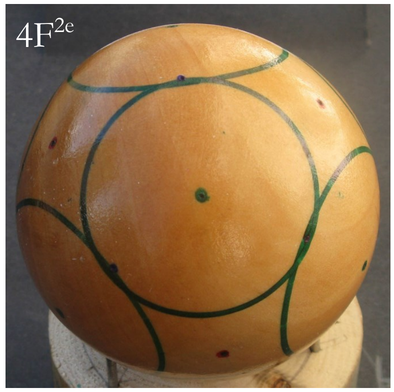
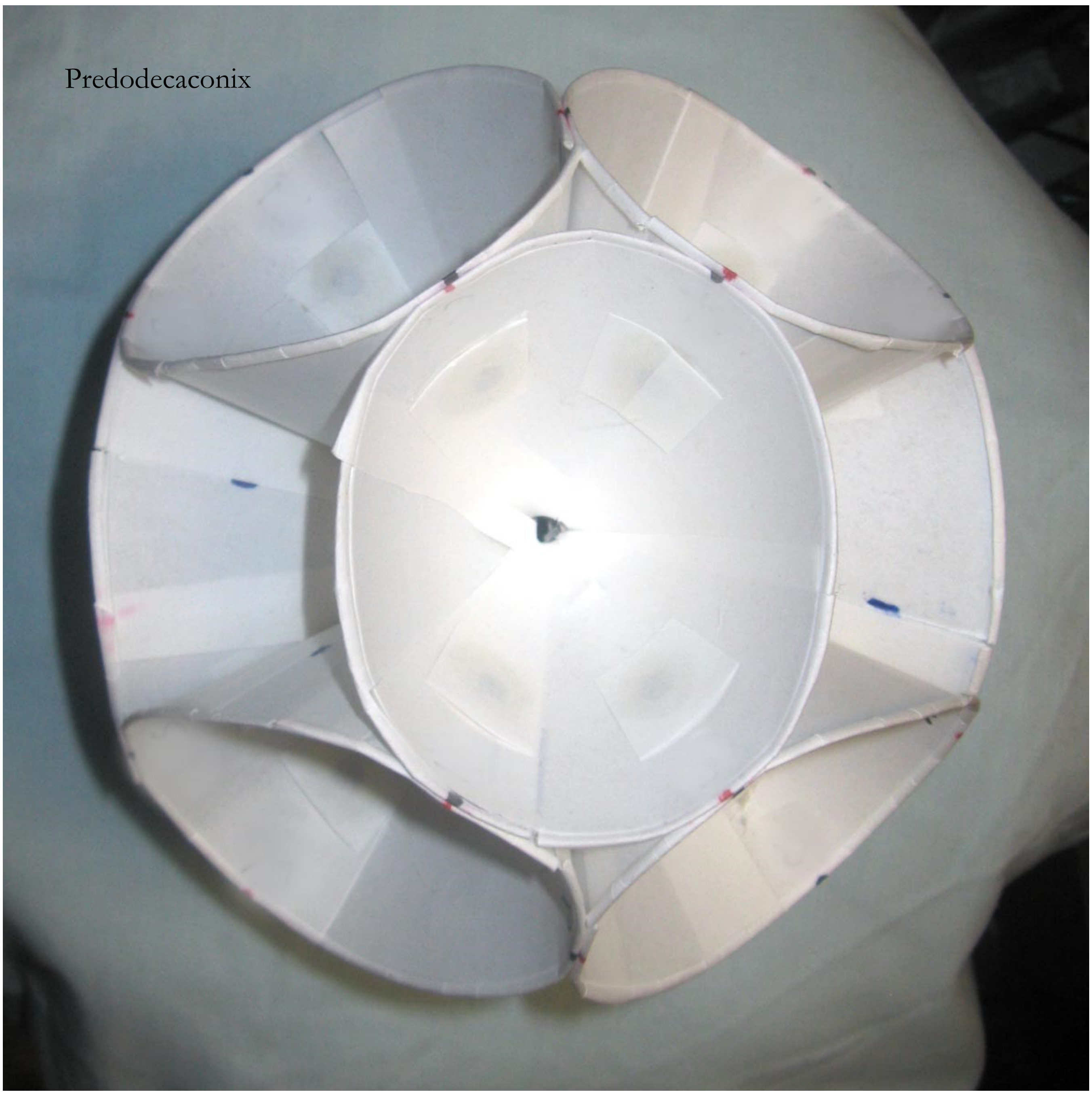


Figure 71

Figure 72



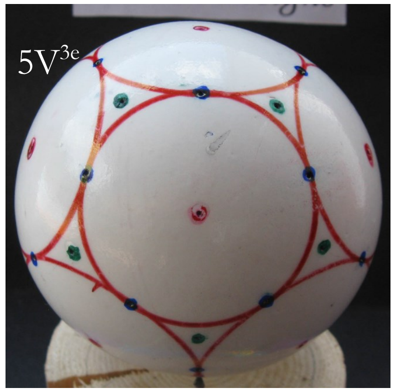


Figure 73

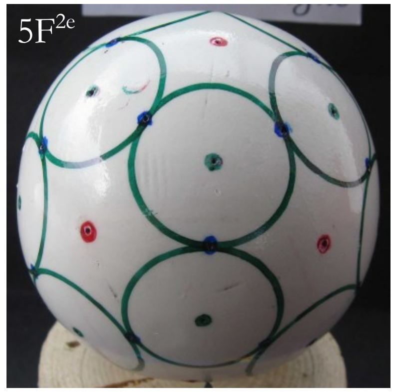
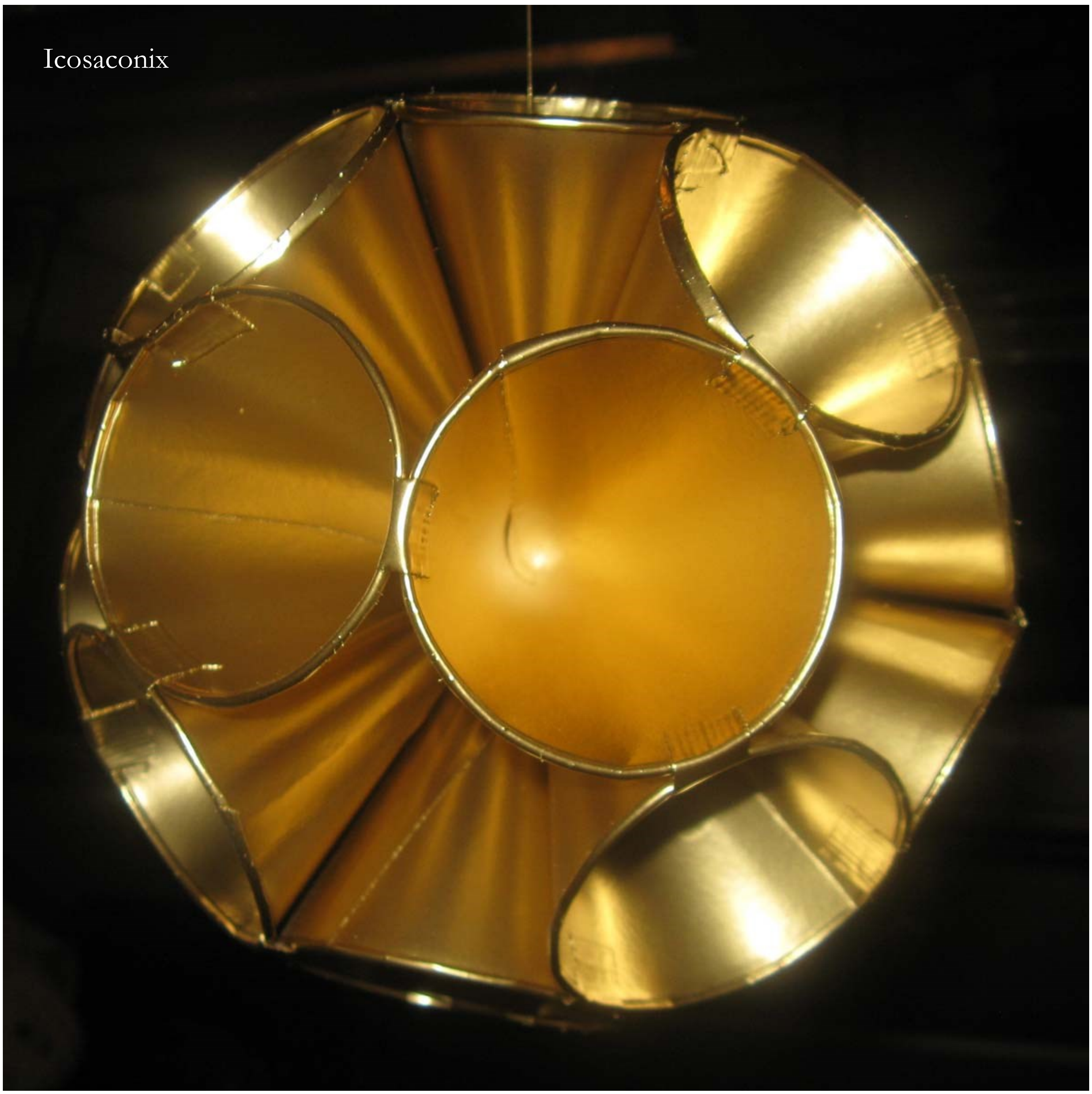


Figure 74



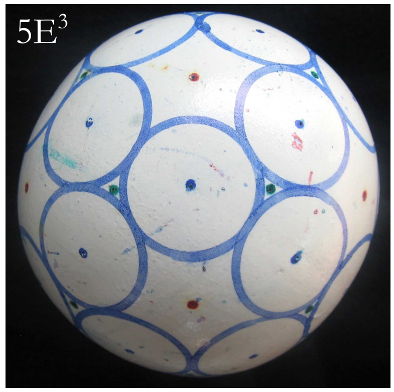


Figure 75

Chapter 6: Fractional Harmonic Notes

The harmonic notes which I have discovered so far have been based on whole integers where the chord angle **w** has the following values:

For 6th harmonic, **w** = $2\pi/6$

For 5th harmonic, **w** = $2\pi/5$

For 4th harmonic, **w** = $2\pi/4$

For 3rd harmonic, **w** = $2\pi/3$

For 2nd harmonic, **w** = $2\pi/2$

This corresponds to $\mathbf{R_{ta}}$ dividing the circumference of its conix into 6, 5, 4, 3, and 2 equal arc segments in one rotation around the circumference of the conix**.** So I started wondering what some of the intermediate fractional values of **w** lead to? I decided to first look between the 2nd and 3rd harmonic to see what I could find. I thought the best place to start looking would be at the halfway point between them at $2\pi/2½$, which I refer to as the 2½ harmonic. I used the general solution from Chapter 2 to calculate the value for R_{ta} for **w** = $2\pi/2½$.

$$(\boldsymbol{E5})\ \boldsymbol{\Omega} = \mathbf{sin^{-1}}(\mathbf{1/2\,sin}\,(\boldsymbol{w/2}))$$

$$(\boldsymbol{E1})\ \boldsymbol{R_{ta}} = \mathbf{2}\boldsymbol{R}\,\mathbf{cos}\,\boldsymbol{\Omega}$$

The resulting value for R_{ta} is approximately equal to 1.701.

Even though I had the calculated value of R_{ta} for the 2½ harmonic conix, I didn't know what pattern, if any, it would lead to. I assumed that some form of geometric tiling would result and that I would be guided by the natural intersection points where to center the next conix. This held true for the Flower of Life and all of the integer based harmonics discovered in the previous chapters. So I started drawing conix on a sphere to see where this would lead. With this length of R_{ta} the conix was drawn deep into the southern hemisphere.

Refer to Figure 26 to see the steps I took to draw the 2½ harmonic note. After drawing the first conix (seen in the center of Figure 26), I made an arbitrary mark M1 on the conix. I moved the compass to M1 and drew the second conix, which intersected the first conix at a point I defined as M2. I moved the compass to M2 and drew the third conix, which intersected the first conix at a point I defined as M3. I moved the compass to M3 and drew the fourth conix, which intersected the first conix at a point I defined as M4. The M4 marking had overshot the starting point M1, which alarmed me at first because the pattern didn't seem to converge on itself as expected. I decided to continue on anyways. I moved the compass to M4 and drew a fifth conix, which intersected the first conix at a point I defined as M5. I moved the compass to M5 and drew a sixth conix, which intersected the first conix at a M1, the starting point and the pattern converged on itself. As you can see in Figure 76, the sequence of compass placements M1 – M2 – M3 – M4 – M5 – M1 required two complete revolutions around the first conix and a five-pointed star is formed by five chords of length R_{ta}. Refer to Figure 77 to see the corresponding 5-petal pattern formed on the top side of the sphere. For a side view of the 2½ harmonic note, see Figure 78. The 2½ harmonic conix is divided into five arc segments in two rotations (5 chords/2 rotations = 2½) around the first conix.

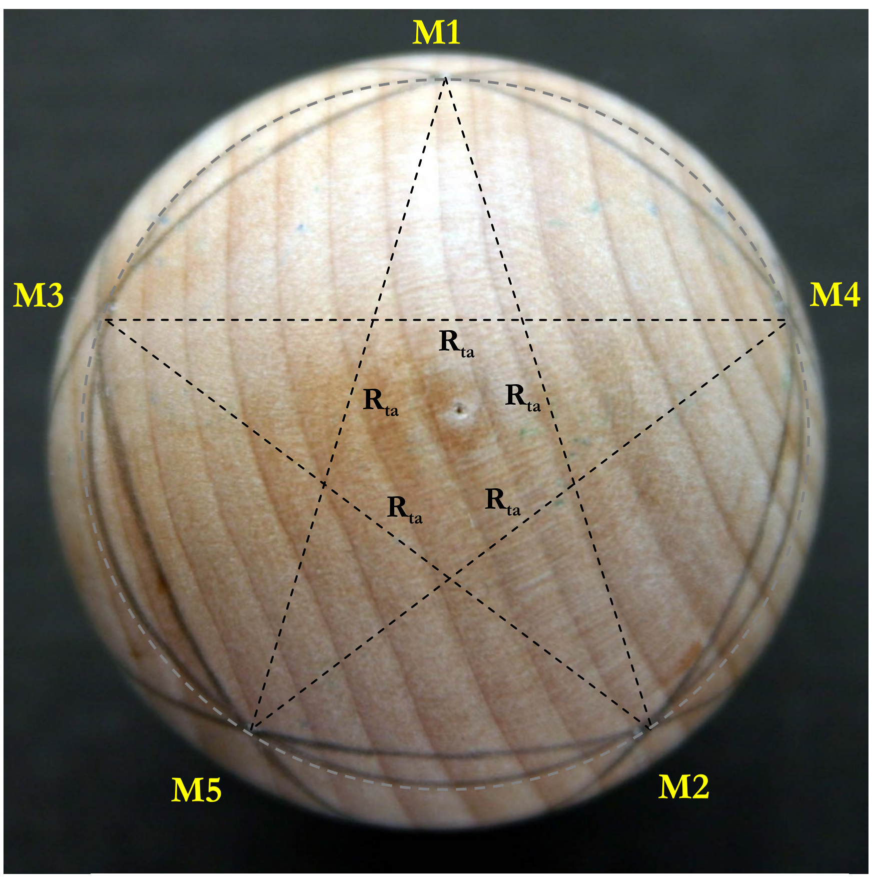


Figure 76 - Bottom view 2½ harmonic

Figure 77 - Top view 2½ harmonic

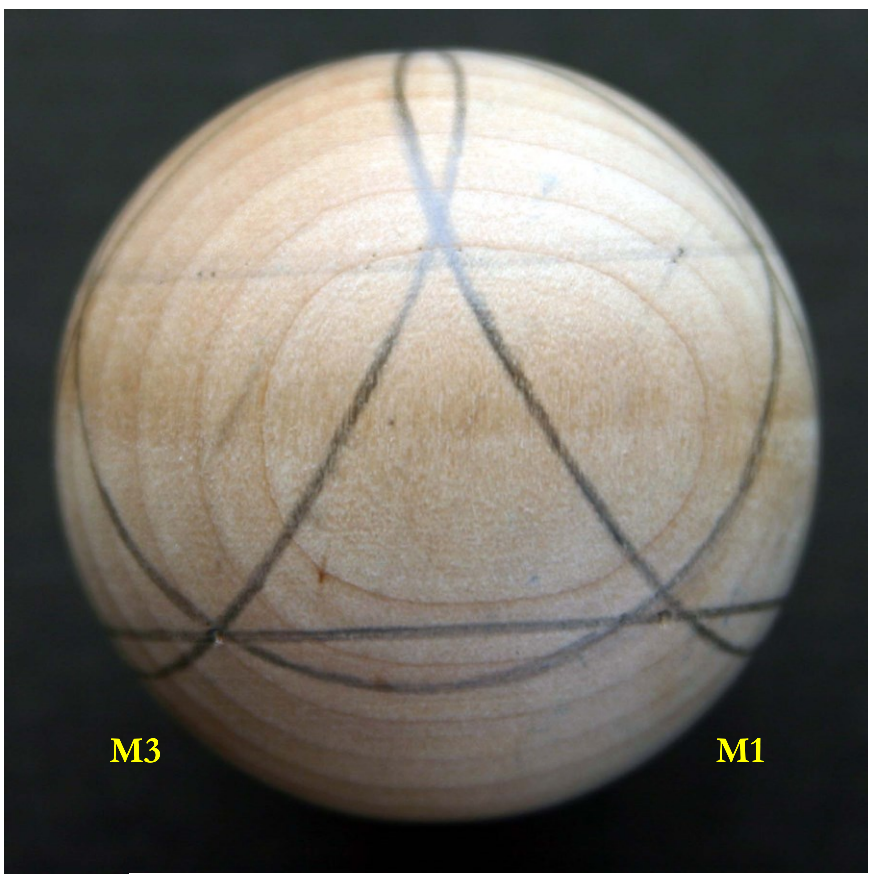


Figure 78 - Side view 2½ harmonic

The next obvious place to look was between the 3rd and 4th harmonic which would be the 3½ harmonic conix. After computing the length of the conical radius R_{ta} I proceeded to construct it. The 3½ harmonic conix is divided into seven arc segments in two rotations (7 chords/2 rotations = 3½). This arrangement of seven conix is interconnected in an overlapping pattern similar to that of a seven-pointed star as seen in Figure 79 created by seven chords of length R_{ta}. The sequence of compass placements M1 – M2 – M3 – M4 – M5 – M6 – M7 – M1 required two complete revolutions around the first conix. It's quite interesting that the first conix is divided into seven equal arc segments because this cannot be done by construction techniques in Euclidean geometry. Refer to Figure 80 to see the corresponding seven-petal pattern formed on the top side of the sphere. For a side view of the 3½ harmonic note, see Figure 81.

You will notice from the side views of these fractional harmonics that they do not fully engulf the sphere as the full integer harmonics did. The harmonic relationship is contained to the first conix and the additional intersection points do not appear to lead to an overlapping pattern. These half-note fractional harmonics require two revolutions around the first conix before it converges into an overlapping pattern. The remaining half-note fractional harmonics (4½ and 5½) are illustrated on the following pages.

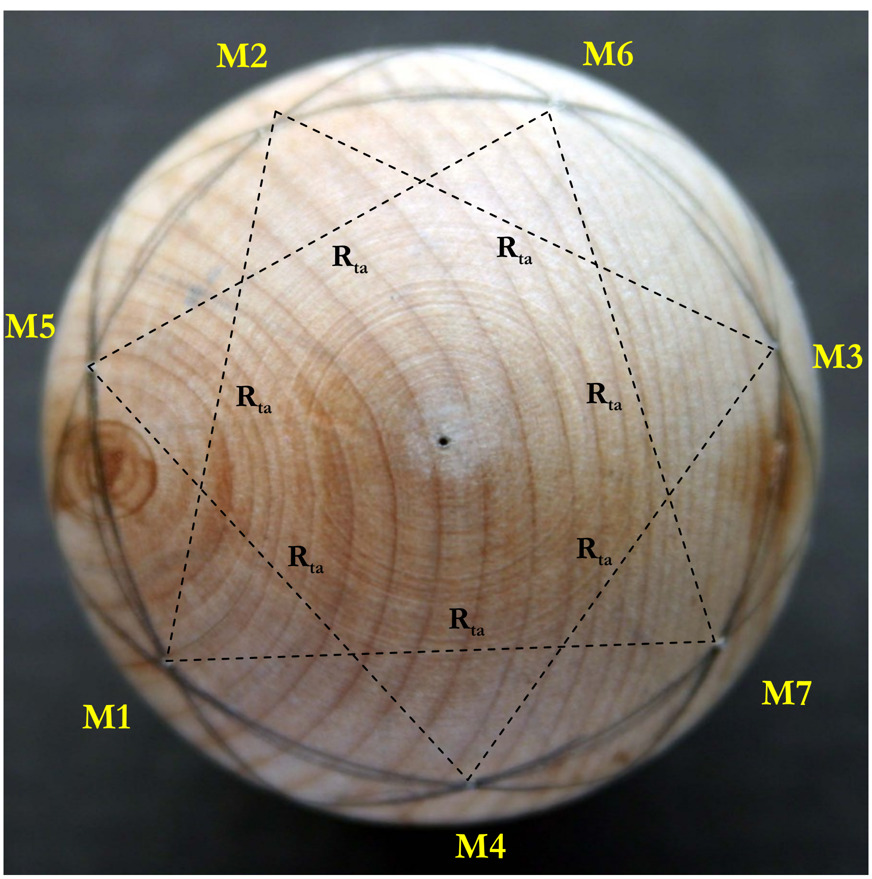


Figure 79 - Bottom view 3½ harmonic

Figure 80 - Top view 3½ harmonic

Figure 81 - Side view 3½ harmonic

Refer to Figure 82 to see how the 4½ harmonic conix is divided into nine arc segments in two rotations (9 chords/2 rotations = 4½). The bottom and side views can be seen in Figure 83-84.

Figure 82 - Top view 4½ harmonic

Figure 83 - Bottom view 4½ harmonic

Figure 84 - Side view 4½ harmonic

Refer to Figure 85 to see how the 5½ harmonic conix is divided into eleven arc segments in two rotations (11 chords/2 rotations = 5½).

This covers all four of the half-note fractional harmonics.

Figure 85 - Top view 5½ harmonic

I suspected there would be more fractional harmonics to find and the next place I looked was at the one third points between the 5^{th} and 6^{th} harmonic note. I wasn't disappointed and as I suspected they required three revolutions before the pattern converged. Refer to Figures 86-89 for illustrations of the 5⅓ harmonic and the 5⅔ harmonic notes.

As seen in Figure 86, the 5⅓ harmonic conix is divided into sixteen arc segments in three rotations (16 chords/3 rotations = 5⅓). Refer to Figure 87 for a side view of the 5⅓ harmonic note.

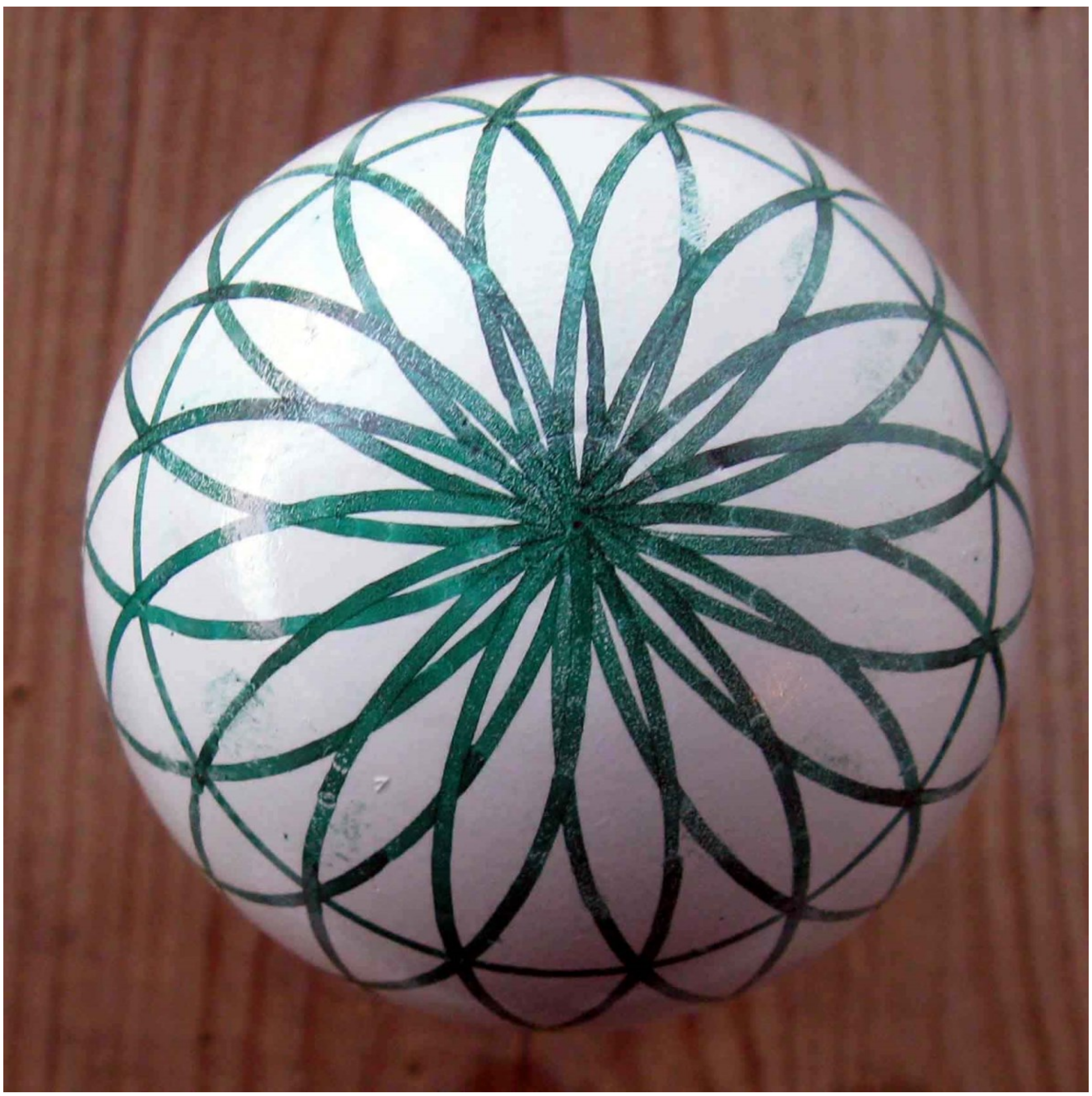


Figure 86 - Top view 5⅓ harmonic

Figure 87 - Side view 5⅓ harmonic

As seen in Figure 88, the 5⅔ harmonic conix is divided into seventeen arc segments in three rotations (17 chords/3 rotations = 5⅔). Refer to Figure 89 for a side view of the 5⅔ harmonic note.

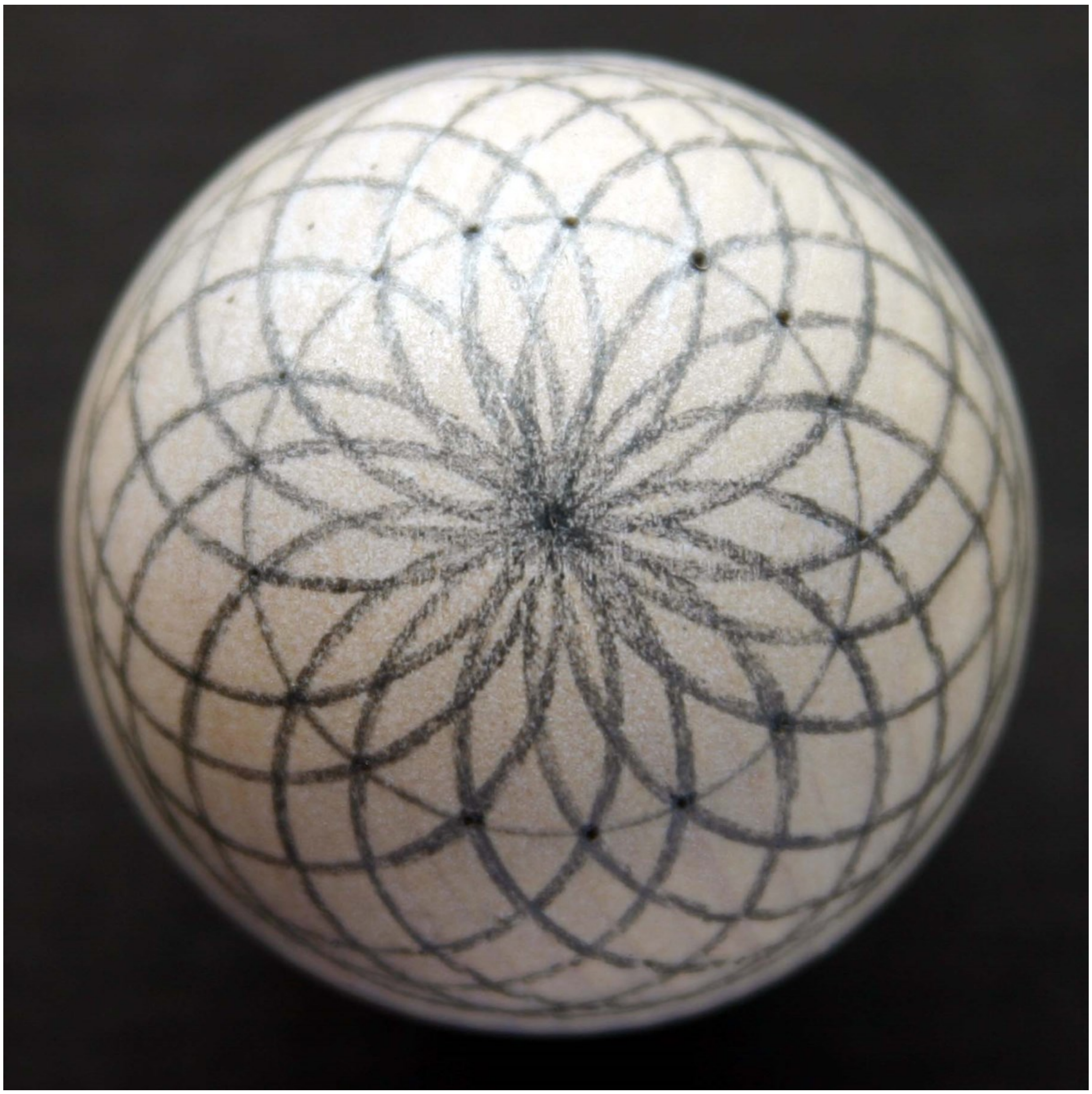


Figure 88 - Top view 5⅔ harmonic

Figure 89 - Side view 5⅔ harmonic

I have cataloged these fractional harmonics in Table VI, in ascending order of the conical radius R_{ta}.

Table VI

6				5		4		3		2	
		5½			4½		3½		2½		
	5⅔		5⅓								

I started to notice a numeric progression in the chord angle **w** used for these fractional harmonic notes. To best represent this numerical progression, the entries in Table VI are inverted and listed in the same order in Table VII. These values represent the chord angle **w/2** which was used in the general solution (E5).

$$(\boldsymbol{E5})\ \boldsymbol{\Omega} = \mathbf{sin^{-1}(1/2\,sin\,(\boldsymbol{w}/2))}$$

Table VII

1/6				1/5		1/4		1/3		1/2	
		2/11			2/9		2/7		2/5		
	3/17		3/16								

The values in Table VII now better describe the unique harmonic properties of each fractional harmonic note. The denominator indicates how many arc segments the conix is divided into, and the numerator indicates how revolutions around the conix are required. The half-note entries (numerator =2) are considered to have a depth of two. The one-third-note entries (numerator =3) are considered to have a depth of three. Refer to Table VIII for a complete collection of harmonics up to a depth of six. I'm not sure that I would have the patience to construct the 5⅚ harmonic. This is table entry 6/35, where the conix is divided into 35 arc segments in six rotations around the conix. However, I could predict its conical radius from the general solution.

Table VIII – List of Fraction Harmonic of a Rotational Depth of 6

#																																		
						6/7																												
					5/6																													
				4/5																														
			3/4																															
		2/3																																
				3/5																														
						4/7																												
								5/9																										
										6/11																								
	1/2																																	
												6/13																						
										5/11																								
								4/9																										
						3/7																												
				2/5																														
							3/8																											
										4/11																								
													5/14																					
																6/17																		
		1/3																																
																		6/19																
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																						6/23												
			1/4																															
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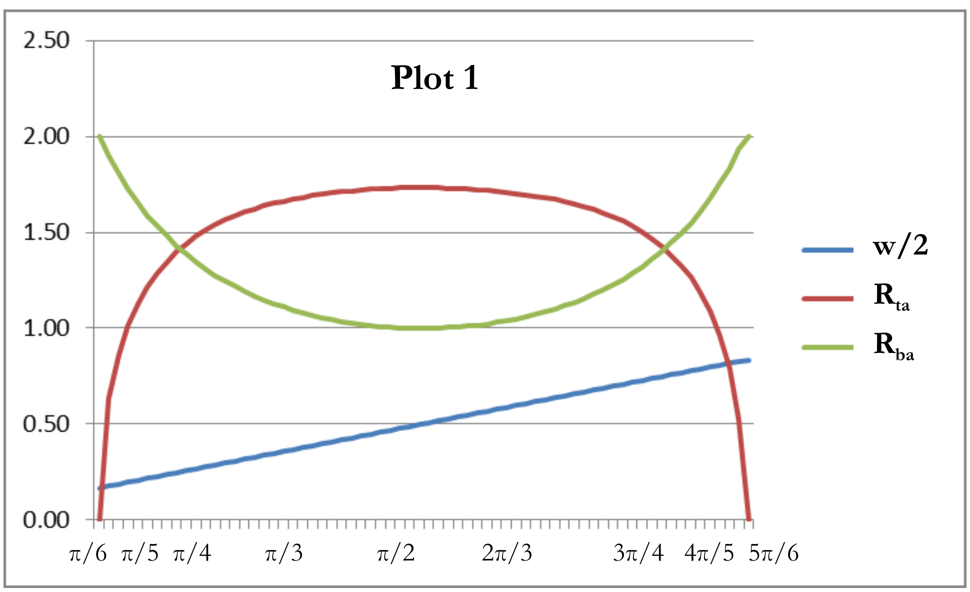
																3/17																		
																						4/23												
																												5/29						
																																		6/35
					1/6																													

Chapter 7: Beyond the 2nd Harmonic

Up to this point I have only explored the harmonic relationships between the 6th and 2nd harmonics, including several fractional harmonics. The chord angle w/2 was limited to the range between π/6 and π/2 and the conical radius R_{ta} had a range between 0 and √3. So what could by lying outside this range? In order to better understand this I decided to plot the general solution for the conical radius R_{ta} versus the chord angle w/2. When I started plotting the values of the chord angles from Table VIII I found that for values larger than 5π/6 the general solution was undefined or invalid. For a chord angle w/2 equal to 5π/6 the conical radius R_{ta} was equal to zero, just as it was for π/6. This established the range of the chord angle w/2 to between π/6 and 5π/6. This can be seen in Plot1. You will notice the symmetry around a chord angle w/2 of π/2. The harmonics that have been described in earlier chapters are the primary harmonics (π/6 - π/2). The harmonics with a chord angle w/2 in the range between π/2 and 5π/6 are the secondary harmonics. . Refer to Table IX to see how the secondary harmonics alias to the primary harmonics.



There is one more category of conix that hasn't been covered yet. All of the conix explored so far has their conical radius in the range between 0 and √3. When the conical radius R_{ta} exceeds a length of √3 (but less than 2) the conix no longer intersect and the harmonic relationship is lost. This is known as the non-harmonic range which is the lower section of the sphere seen in Figure 90. I have attempted to draw a few of these non-harmonic arrangements of conix however since the conix are not directly coupled no immediate pattern was visible. I hope explore this in more detail at a later time.

Table IX

Primary Harmonic **Secondary Harmonic**

Harmonic	Chord Angle	Chord Angle	Harmonic
6	$\pi/6$	$5\pi/6$	1⅕
5½	$2\pi/11$	$9\pi/11$	$1^2/_9$
5	$\pi/5$	$4\pi/5$	1¼
4½	$2\pi/9$	$7\pi/9$	$1^2/_7$
4	$\pi/4$	$3\pi/4$	1⅓
3½	$2\pi/7$	$5\pi/7$	1⅖
3	$\pi/3$	$2\pi/3$	1½

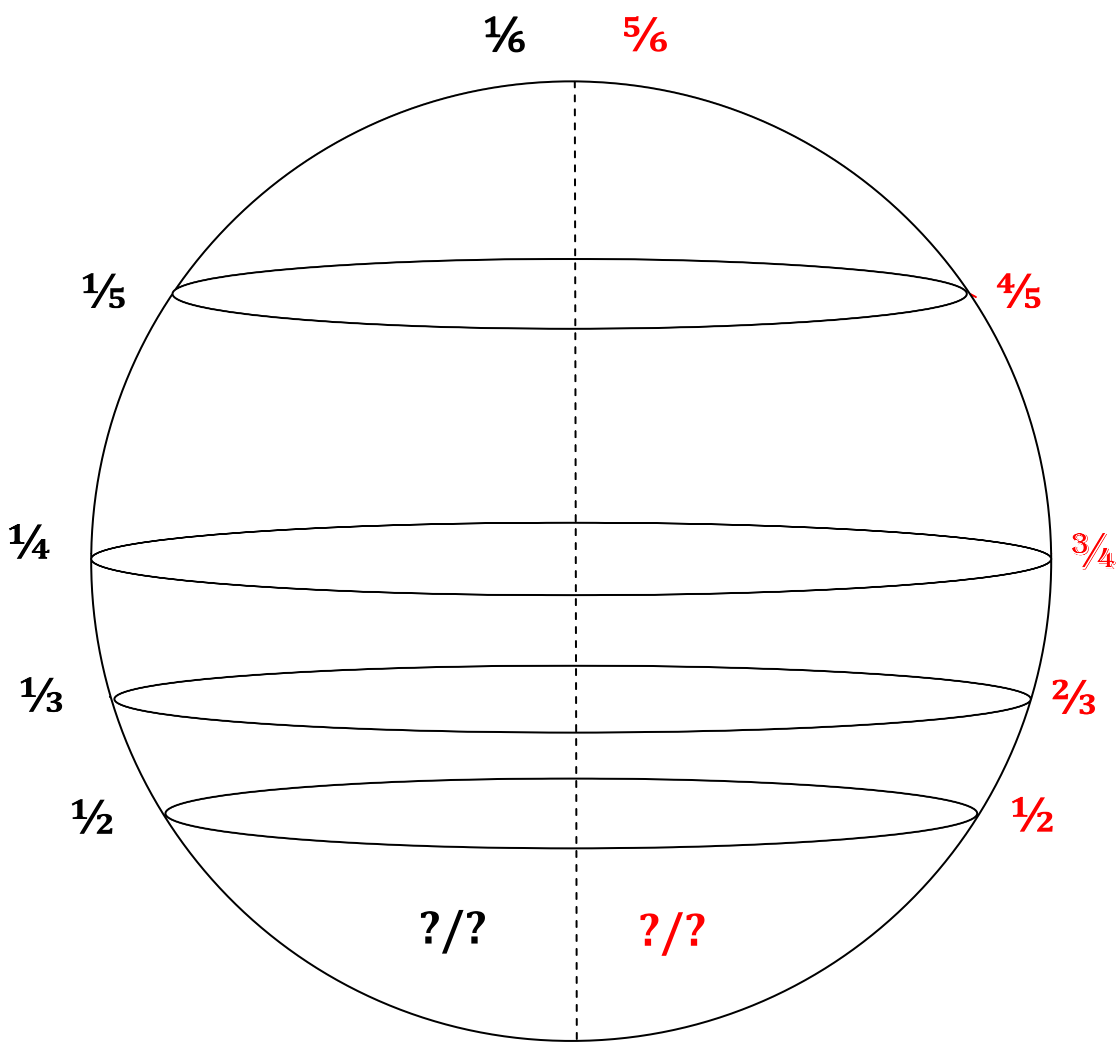


Figure 90

Chapter 8: Relationship between the Antipodal Conical Radii

Geometrically, there is a straight forward relationship between both conical radii R_{ta} and R_{ba} as seen in Figure 1 of Chapter 1.

$$4R^2 = (R_{ta})^2 + (R_{ba})^2$$

However, by and in large they each have a separate geometric relationship with the conix and sphere. I found this out first hand when I tried using conical radius R_{ba} back in Chapter 2 to construct the 3rd harmonic note and the conix didn't converge. I happened to find a few exceptions to this which I will describe.

Sometime after constructing the 2½ harmonic, I started to inspect it in more detail and noticed that there were some conix intersection points that I could have used to center new conix, refer to Figure 91. In fact I probably didn't even look for these intersection points because from past experience of drawing fractional harmonics, these additional conix didn't converge. However, in this case these new intersection points created another 2½ harmonic on the south side of the sphere and in doing so actually formed the identical pattern of a 5th harmonic note, refer back to Figure 10 in Chapter 1. Refer to Figure 92-94 for a composite view of the combination of two 2½ harmonic notes (one colored red and one colored green). I did not expect to see this serendipitous result which equates to one 2½ harmonic plus another 2½ harmonic equals a 5th harmonic note (2½ + 2½ = 5).

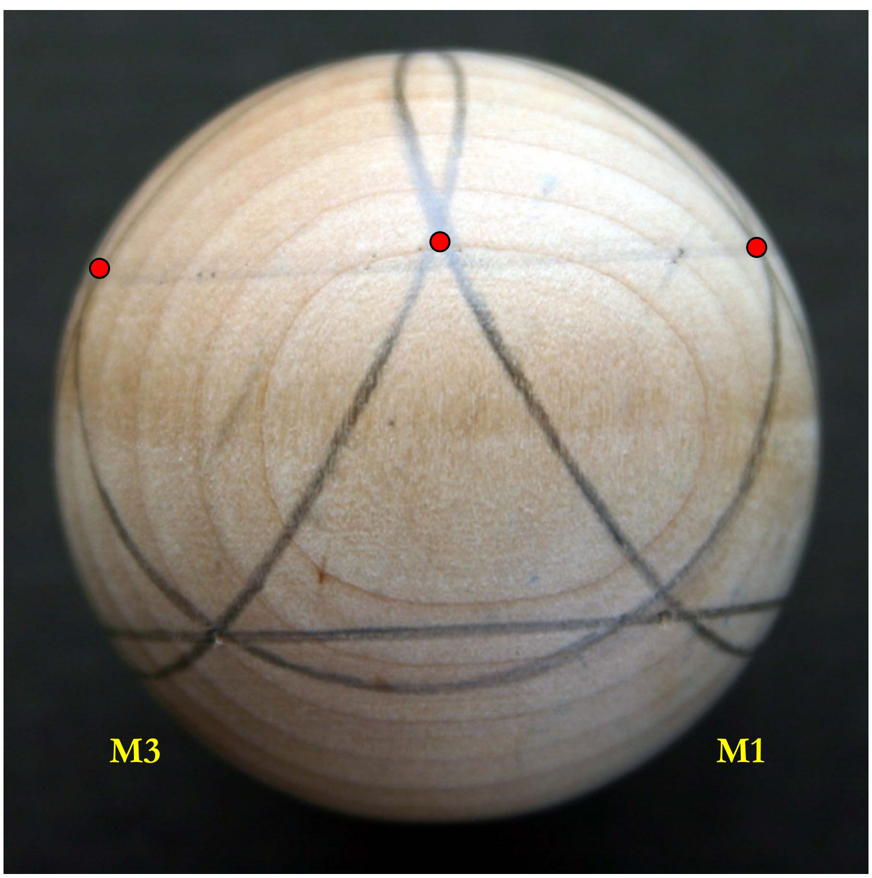


Figure 91 - Side view 2½ harmonic

In Figure 92 you can see a top view of one green 2½ harmonic.

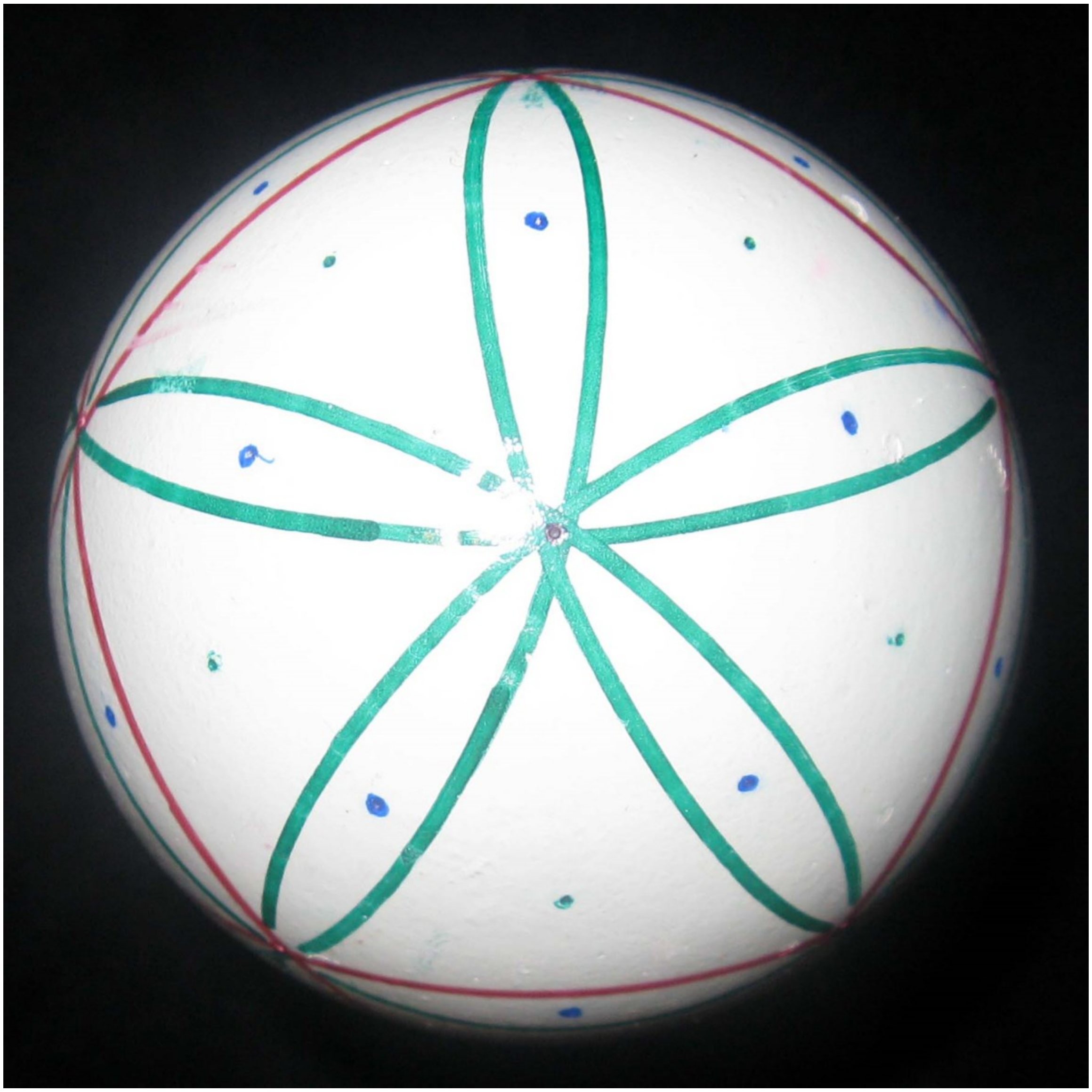


Figure 92 – Top view 2½ + 2½ harmonic

In Figure 93 you can see a side view with the green 2½ harmonic on the top and a red 2½ harmonic on the bottom.

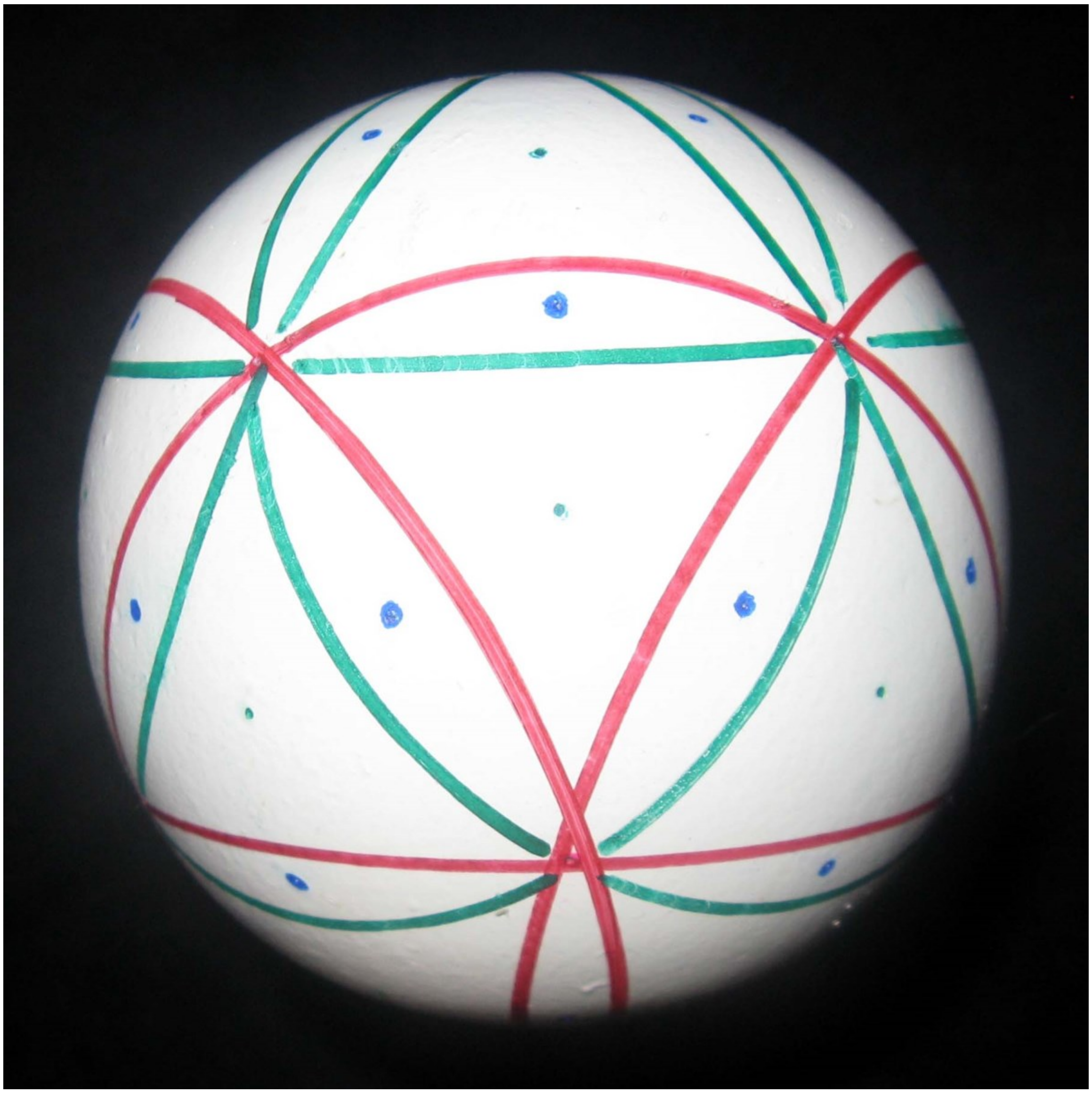


Figure 93 – Side view 2½ + 2½ harmonic

In Figure 94 you can see a bottom view of one red 2½ harmonic.

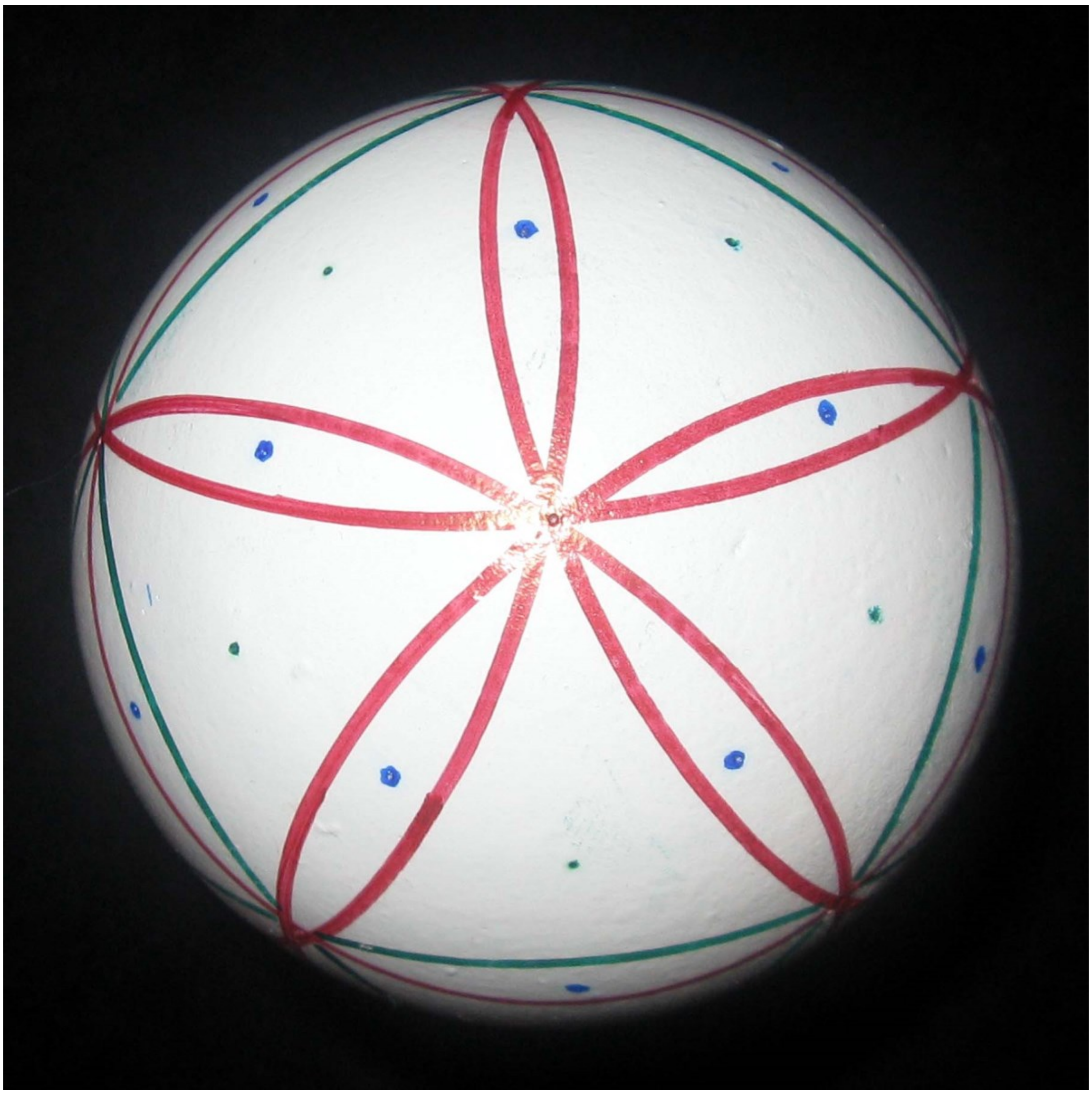


Figure 94 – Bottom view 2½ + 2½ harmonic

I now realize the mistake I had made when I first drew the 2½ harmonic. I had drawn many other fractional harmonics prior to the 2½ harmonic and when I tried centering new conix on the other intersection points, there was no pattern convergence. For example in the 5⅓ harmonic (refer back to Figure 87) there were 112 intersection points and when I attempted to add new conix the pattern started to get quite messy. That being said, I wrongly made the assumption for the 2½ harmonic that the other intersection points would not lead to anything.

Then just by chance, I had noticed that the value for the conical radius R_{ta} of the 2½ harmonic was equal to the conical radius R_{ba} of the 5^{th} harmonic. What is surprising is that this is the only pair of harmonics that I could find that have this unique relationship (i.e. R_{ba} of one harmonic equal R_{ta} of another). It turns out that the conix of the 2½ harmonic and the 5^{th} harmonic are the same size (i.e. their planar radius R_p are equal), but the way they are drawn is quite different. For the 5^{th} harmonic, the compass was placed on the north pole and the conix was drawn in the northern hemisphere. While for the 2½ harmonic, the compass was placed on the north pole and the conix was drawn in the southern hemisphere and it took two revolutions before the conix converged, yet the pattern was identical.

I found one other unexpected relationship as I was analyzing other harmonics to see what would happen if I used the conical radius R_{ba} instead of R_{ta}. The 4^{th} harmonic conix was the first placed I looked but quickly realized that for this conix, R_{ta} was equal to R_{ba} so there was nothing to be found here. I did spend some time investigating the 3^{rd} harmonic, but the use of conical radius R_{ba} did not result in any pattern convergence.

The two conical radii of the 2^{nd} harmonic note have a unique relationship between themselves. Conical radius R_{ta} is equal to $\sqrt{3}$ and R_{ba} is equal to 1. Three new conix of conical radius R_{ba} can be added to the 2^{nd} harmonic note. These new conix are centered at the point where two conix abut in the 2^{nd} harmonic note and interleave perfectly, refer to Figure 95-96. It immediately struck me that these six conix divide the sphere's equator into six equal arc segments.

Figure 95

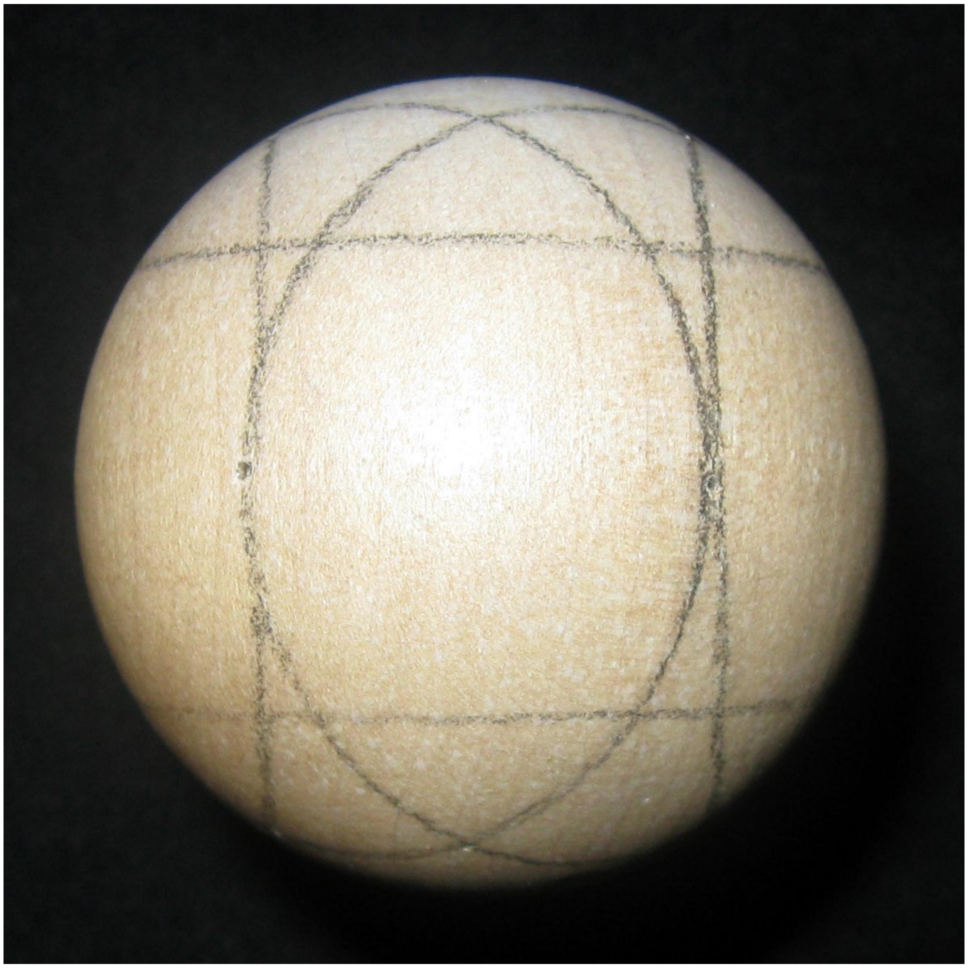


Figure 96

At first this didn't seem that significant. In fact, it wasn't until the next day that I saw how significant this was. I was drawing on a unit sphere (one inch radius R) and I was using a unit conix (conical radius equal to one inch) to divide the equator into six equal arc segments. This is analogous to the relationship seen in Euclidean geometry where a unit circle is divided into six equal arc segments by its radius which is equal to 1. In spherical geometry, a unit sphere's equator is divided into six equal arc segments by a unit conix where its conical radius is equal to 1.

In addition to this, a smaller conix of planar radius r_p equal to ½ fits perfectly in the spherical hexagon created around the north and south poles. The corresponding conical radius of these smaller conix are r_{ta} equal to $\sqrt{(2-\sqrt{3})}$ and r_{ba} equal to $\sqrt{(2+\sqrt{3})}$. Refer to Figure 97.

$$r_p = {}^{1}/_{2} \qquad \Omega = {}^{\pi}/_{12}$$

$$r_{ta} = \sqrt{2+\sqrt{3}} \qquad \theta = {}^{5\pi}/_{6}$$

$$r_{ta} = \sqrt{2-\sqrt{3}} \qquad \phi = {}^{5\pi}/_{12}$$

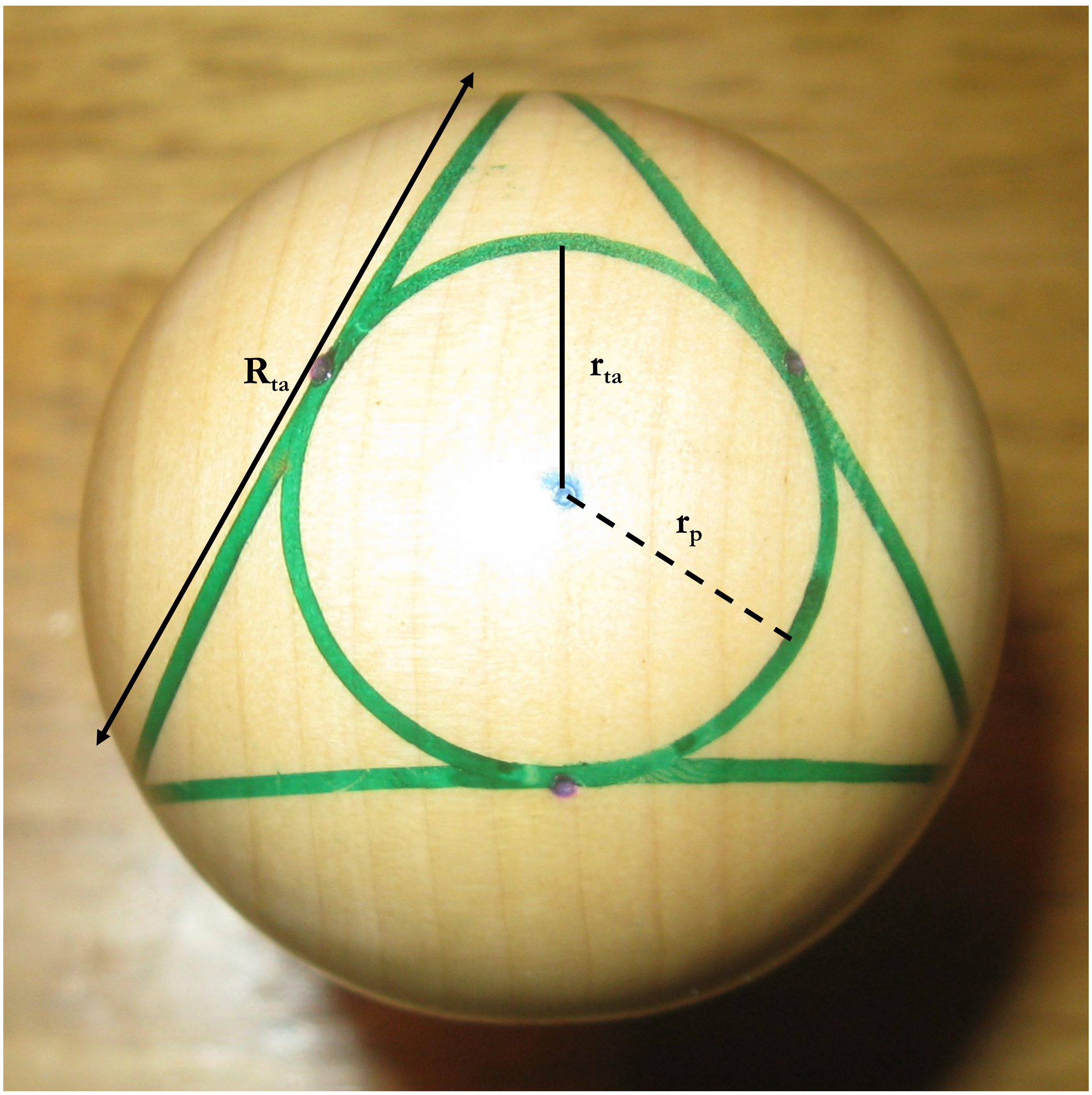


Figure 97

Appendix I: Harmonic Chords Created from Harmonic Notes

Numerous harmonic notes have been illustrated in the previous chapters. This provides a rich selection to choose from to compose what is referred to as a harmonic chord, which is a combination of two or more harmonic notes. I have not yet attempted to compose harmonic chords from different harmonics, but I suspect some compositions are possible.

With twenty three harmonic notes to choose from in the 5th harmonic alone results in millions of possible harmonic chord compositions. Only a few have been illustrated on the following pages.

Other than its innate geometric beauty, there is nothing more to be said about harmonic chords. Perhaps sometime in the future more will be discovered regarding these harmonic chords.

The following notation represents a 5th harmonic chord with note **5V** and **5V^{2f}**. Refer to Figure 98 for view that is aligned with the vertices of the icosahedron.

[**5V:5V^{2f}**]

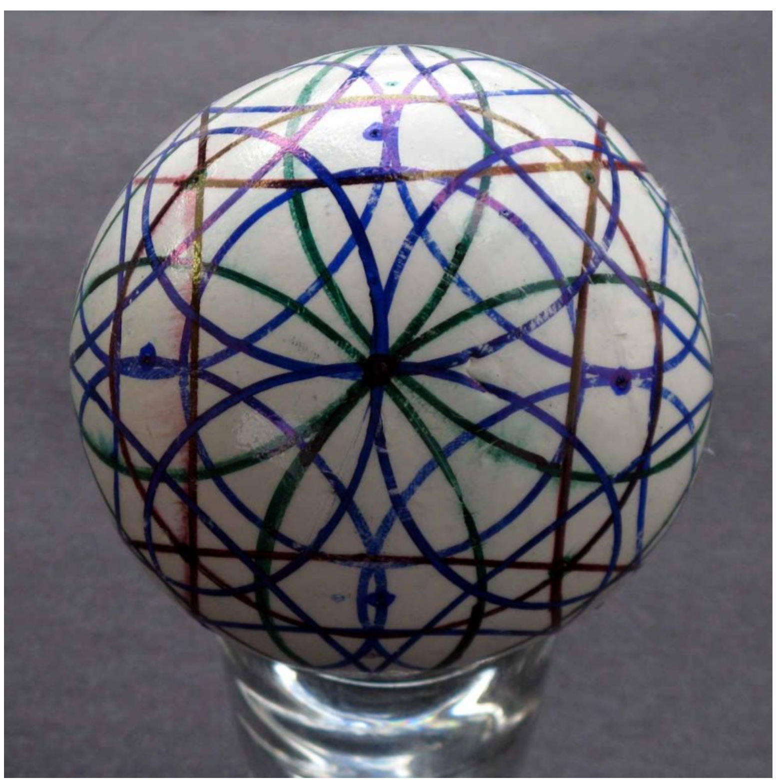
Figure 98

The following notation represents a 5th harmonic chord with notes **5V**, **5V**^{2f} and **5F**^{2e}. Refer to Figure 99 for a view that is aligned with the vertices of the icosahedron.

[5V:5V^{2f}:5F^{2e}]

Figure 99

Here is a random assortment of some other harmonic chords.



Appendix II: Harmonic Scales and Ratios

Given the rich collection of conix in the various 5^{th} harmonic intervals, I was curious what relationships may exist between them. I started by placing each interval of conix in a co-centric arrangement on a sphere, as seen in Figure 100-102. A composite image of all intervals (V-note, F-note and E-note) can be seen in Figure 103. There was no obvious pattern evident at first inspection. The pattern did not appear to be random and there was some grouping or clustering among the conix.

Using a compass, I measured the conical radius of each conix and put them in a spreadsheet for a quick analysis. I programmed the spreadsheet to perform various comparisons of the conix. I had already discovered a harmonic relationship between the conix and sphere for each harmonic note, so perhaps there was some harmony between the conix of different notes. My first hunch was to look for ratios that would be similar to that of a music scale, of which there are many. I knew my measurements were imprecise, but I felt they were accurate enough to detect a trend.

To extract the ratios, I would set one conix as the tonic (ratio of 1). Then I would divide the next smaller conix by the tonic, and so on, just as would be done for a musical scale. The results were a little inconclusive, but there was a hint or suggestion of some fragments of ratios which were similar in nature to those found in a diatonic or chromatic scale.

This piqued my interest, but I knew that in order to extract the exact ratios I would need to derive the geometric properties of each conix. This seemed a little daunting at first, but I thought I had enough information to proceed. The actual geometric derivations can be found in my book "Sphere of Life VI" in Appendix VII.

Figure 100

Refer to Figures 101 for a view of the 5^{th} harmonic F-interval.

Figure 101

Refer to Figures 102 for a view of the 5th harmonic E-interval.

Figure 102

Refer to Figures 103 for a composite view of all three 5^{th} harmonic intervals.

Figure 103

Now I had some raw, but precise ϴ-based data to work with. Since I didn't know where to start looking, I calculated the ratios of every pair of conix and searched the results for simple ratios in this sea of complexity.

To my amazement, a few simple ratios did surface which were small integer fractions of π. The conix involved were all contained in the E-interval [**5E**$_{6e}$ – **5E**$_{3e}$ – **5E**$_{1e}$ – **5E**]. If I set **5E**$_{6e}$ as the tonic conix, I get the following angle-based scale (ϴ-scale). See Table X.

Table X - ϴ and Ω Scales

V Interval	H Interval	E Interval	ϴ	ϴ-scale		Ω	Ω-scale
5V$_{2e}$	**5F$_{5e}$**	**5E$_{6e}$**	**$\pi/2$**	**1**		**$\pi/4$**	**5:8**
5V$_{1f}$	5F$_{4v}$						
		5E$_{5e}$	**$2\pi/5$**	**4:5**		**$3\pi/10$**	**3:4**
	5F$_{3f}$						
	5F$_{2e}$	5E$_{4f}$					
5V							
		5E$_{3e}$	**$\pi/3$**	**2:3**		**$\pi/3$**	**5:6**
5V^{1e}		5E$_{2v}$					
	5F$_{1e}$	5E$_{1f}$					
	5F						
5V^{2f}	5F^{1v}						
		5E	**$\pi/5$**	**2:5**		**$2\pi/5$**	**1**
5V^{3e}		5E^{1v}					
	5F^{2e}	5E^{2f}					

Here's an example of a "just interval" (or just intonation, in which the frequencies of notes are related by ratios of small whole numbers). Notice some identical ratios from the ϴ-scale:

[1 – 15:16 – 8:9 – 5:6 **– 4:5 – 3:4 – 32:45 – 2:3 –** 5:8 **– 3:5 –** 9:16 **– 8:15 - 1:2]**

[C D^b D E^b E F G^b G A^b A B^b B C_2]

Here's how this ϴ-scale would look in the key of C based on the "just interval" seen above. I have assigned the corresponding notes from the interval to each ϴ-scale ratio as seen in red. Three of these E-interval ratios were an exact match of the ratios found in a "just interval".

[1 - 4:5 – 2:3 – 2:5**]**

[C E G E_2]

Now I will take the same four conix above, and instead of using the spherical angle ϴ, I would use the corresponding conical angle Ω and extract another set of ratios. See Table X. This resulted in the following scale (Ω-scale).

Here's what the Ω-scale would look in the key of C based on the on the "just interval" seen above. I have assigned the corresponding notes from this interval to each Ω-scale ratio.

[5:8 – 3:4 – 5:6 **– 1]**

[A^b F E^b C]

This was another direct hit on four of the above musical scale ratios. I noticed that the ϴ-scale and Ω-scale dovetail together nicely and form the following combined scale (ϴΩ-scale). I did take some liberty in dovetailing these two scales. Perhaps it's just a striking coincidence, only the future will tell.

[1 – 15:16 – 8:9 – 5:6 **– 4:5 – 3:4 – 32:45 – 2:3 –** 5:8 **– 3:5 –** 9:16 **– 8:15 - 1:2]**

[C D^b D E^b E F G^b G A^b A B^b B C_2]

The Ω-scale and ϴ-scale contain some of the exact ratios found in the "just interval"

I should point out that the similarities between the ratios of conix and those in a music scale are only my observations. I don't know if this infers a deeper relationship between conix and music, or if this is just a coincidence.

The angle based scales (Θ-scale and Ω-scale) were simple ratios of the spherical and conical angles of conix from the E-interval. I also discovered some other ratios that have a slightly more complicated relationship which can be represented by simple trigonometric equations, but they were elusive!

By deriving cosines of the various conix and extracting their ratios, simplicity surfaces again. These ratios can be described as simple fractions of small integers and square roots. This cosine-scale is captured in Table XI along with the sin-scale and tan-scale in Table XII-XIII. Discovering these ratios was made possible because I had derived the spherical angle for each conix as simple trigonometric values.

The transformation function used to create these ratios is described in the following notation:

$$\cos(5F_{3f}(\Theta))$$

This is the cosine of the spherical angle of the 5th harmonic F-note $5F_{3f}$.

I must confess I have no idea if there is any significance to these ratios, but in a sea of complexity it is nice to see some simplicity surface. To be honest, I don't know what led me to discover this relationship.

Table XI - Cosine Scale

V Interval	H Interval	E Interval	Θ	cos-scale
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$		
$5V_{1f}$	$5F_{4v}$			
		$5E_{5e}$		
	$5F_{3f}$		$\cos^{-1}(1/3)$	$1/\sqrt{5}$
	$5F_{2e}$	$5E_{4f}$		
5V			$\cos^{-1}(1/\sqrt{5})$	$3/5$
		$5E_{3e}$	$\cos^{-1}(1/2)$	$3/2\sqrt{5}$
$5V^{1e}$		$5E_{2v}$		
	$5F_{1e}$	**$5E_{1f}$**	$\cos^{-1}(1/\sqrt{3})$	$\sqrt{3}/\sqrt{5}$
	5F		$\cos^{-1}(\sqrt{5}/3)$	1
$5V^{2f}$	$5F^{1v}$			
		5E		
$5V^{3e}$		$5E^{1v}$		
	$5F^{2e}$	$5E^{2f}$		

Table XII - Sine Scale

V Interval	H Interval	E Interval	Θ	sin-scale
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$		
$5V_{1f}$	$5F_{4v}$			
		$5E_{5e}$		
	$5F_{3f}$		$\sin^{-1}(2\sqrt{2}/3)$	1
	$5F_{2e}$	$5E_{4f}$		
5V			$\sin^{-1}(2/\sqrt{5})$	$\sqrt{3}/\sqrt{2}\sqrt{5}$
		$5E_{3e}$	$\sin^{-1}(\sqrt{3}/2)$	$3\sqrt{3}/4\sqrt{2}$
$5V^{1e}$		$5E_{2v}$		
	$5F_{1e}$	$5E_{1f}$	$\sin^{-1}(\sqrt{2}/\sqrt{3})$	$\sqrt{3}/2$
	5F		$\sin^{-1}(2/3)$	$1/\sqrt{2}\sqrt{3}$
$5V^{2f}$	$5F^{1v}$			
		5E		
$5V^{3e}$		$5E^{1v}$		
	$5F^{2e}$	$5E^{2f}$		

There were two tangent scale discovered. One was based on simple fractions and roots, while the other was based on the golden mean **ψ.**

Table XIII - Tangent Scale

V Interval	H Interval	E Interval	ϴ	tan-scale	ψ-scale
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$			
$5V_{1f}$	$5F_{4v}$				
		$5E_{5e}$			
	$5F_{3f}$		$\tan^{-1}(2\sqrt{2})$	1	
	$5F_{2e}$	$5E_{4f}$	$\tan^{-1}(\varphi+1)$		1
5V			$\tan^{-1}(2)$	$1/\sqrt{2}$	
		$5E_{3e}$	$\tan^{-1}(\sqrt{3})$	$\sqrt{3}/2\sqrt{2}$	
$5V^{1e}$		$5E_{2v}$	$\tan^{-1}(\varphi)$		$\psi-1$
	$5F_{1e}$	$5E_{1f}$	$\tan^{-1}(\sqrt{2})$	$1/2$	
	5F		$\tan^{-1}(2/\sqrt{5})$	$1/\sqrt{2}\sqrt{5}$	
$5V^{2f}$	$5F^{1v}$		$\tan^{-}(2(\varphi-1)/\varphi)$		$2/(\psi+1)^2$
		5E			
$5V^{3e}$		$5E^{1v}$	$\tan^{-1}(\varphi-1)$		$(\psi-1)/(\psi+1)$
	$5F^{2e}$	$5E^{2f}$	$\tan^{-1}((\varphi-1)/\varphi)$		$1/(\psi+1)^2$

I'm not sure how I came across the following scale, it just kind of surfaced from my number crunching efforts and I recognized something peculiar. Instead of using the angle based value of the conix **5V**(ϴ), I computed the length of the conical radius for R_{ta} **5V**(R_{ta}), and then I extracted the length-based ratios. No simple fraction based ratios could be found. However it turned out that the ratios could be described as simple functions instead of a fractional ratio. These ratios were equivalent to the **sine** of the spherical angle ϴ of some other conix. This ratio could be described with the following notation:

$$\boldsymbol{5F}_{2f}(R_{ta})/\boldsymbol{5E}_{5e}(R_{ta}) = \sin\left(\boldsymbol{5V}_{1f}(\theta)\right)$$

Refer to the following Table XIV for the other conix in this category. I also found a few other traces of similar ratios of other conix, but I had reached the limit of effectiveness of my number crunching technique and I would need to fine tune it to find more ratios. I'll address this more detail in a future publication. I suspect that there are several more undiscovered/hidden ratios yet to be found.

Table XIV - Complex Scale

V Interval	H Interval	E Interval	R_{ta} Ratio	Scale
$5V_{2e}$	$5F_{5e}$	$5E_{6e}$		
$5V_{1f}$	$5F_{4v}$			
		$5E_{5e}$	$5E_{5e}(R_{ta})/5E_{5e}(R_{ta})$	1
	$5F_{3f}$		$5F_{3f}(R_{ta})/5E_{5e}(R_{ta})$	$\sin(5V_{1f}(\Theta))$
	$5F_{2e}$	$5E_{4f}$		
5V			$5V(R_{ta})/5E_{5e}(R_{ta})$	$\sin(5V(\Theta))$
		$5E_{3e}$	$5E_{3e}(R_{ta})/5E_{5e}(R_{ta})$	$\sin(5V^{1e}(\Theta))$
$5V^{1e}$		$5E_{2v}$		
	$5F_{1e}$	$5E_{1f}$		
	5F		$5F(R_{ta})/5E_{5e}(R_{ta})$	$\sin(5V^{2f}(\Theta))$
$5V^{2f}$	$5F^{1v}$			
		5E	$5E(R_{ta})/5E_{5e}(R_{ta})$	$\sin(5V^{3e}(\Theta))$
$5V^{3e}$		$5E^{1v}$		
	$5F^{2e}$	$5E^{2f}$		

Appendix III: Incremental Sub-Notes within Harmonic Intervals

I have also explored making some slight alterations to some of the harmonic notes. See Figure 104-106 for examples of a half step between **5V** and **5V**$_{1f}$ viewed from the vertex, edge and face respectively.

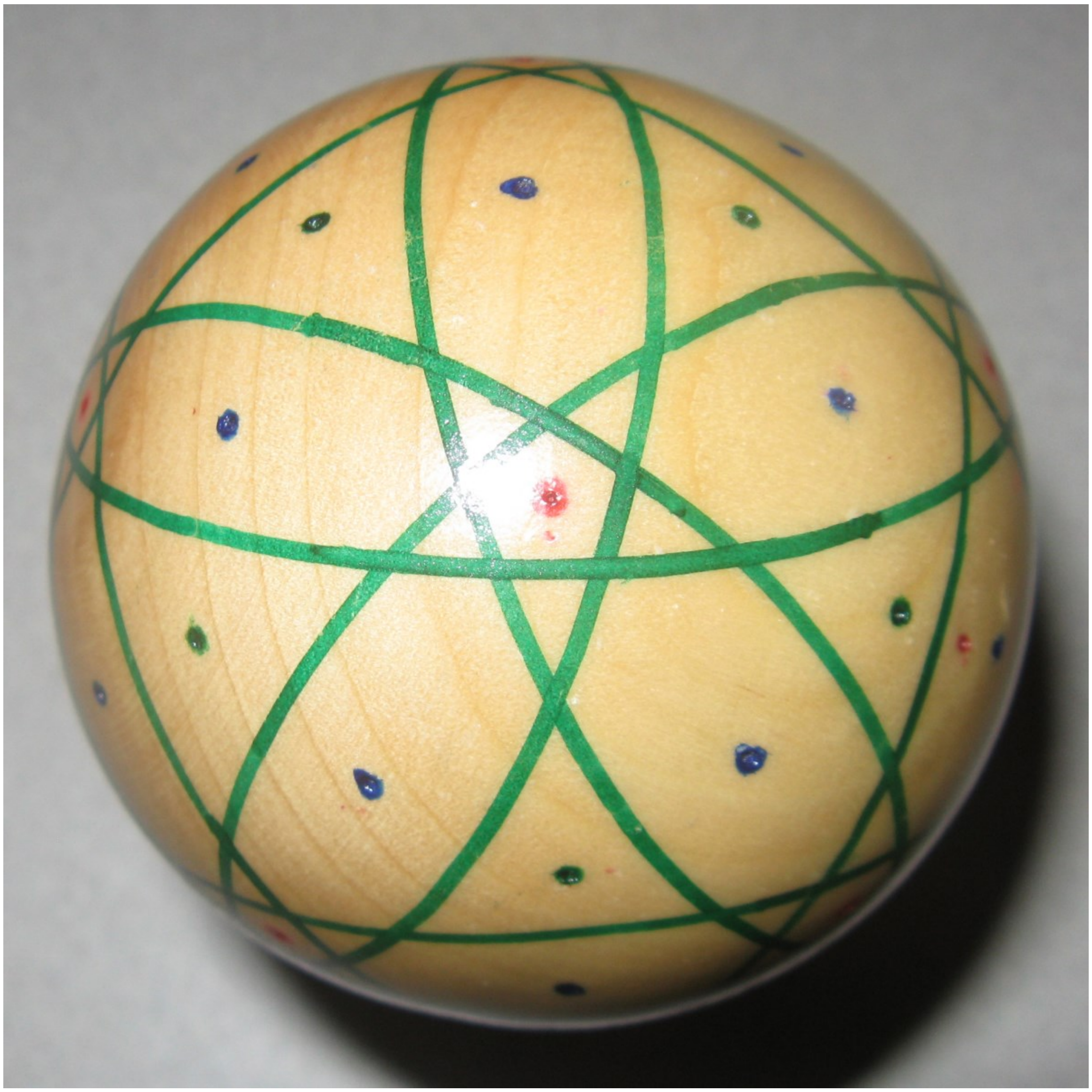


Figure 104

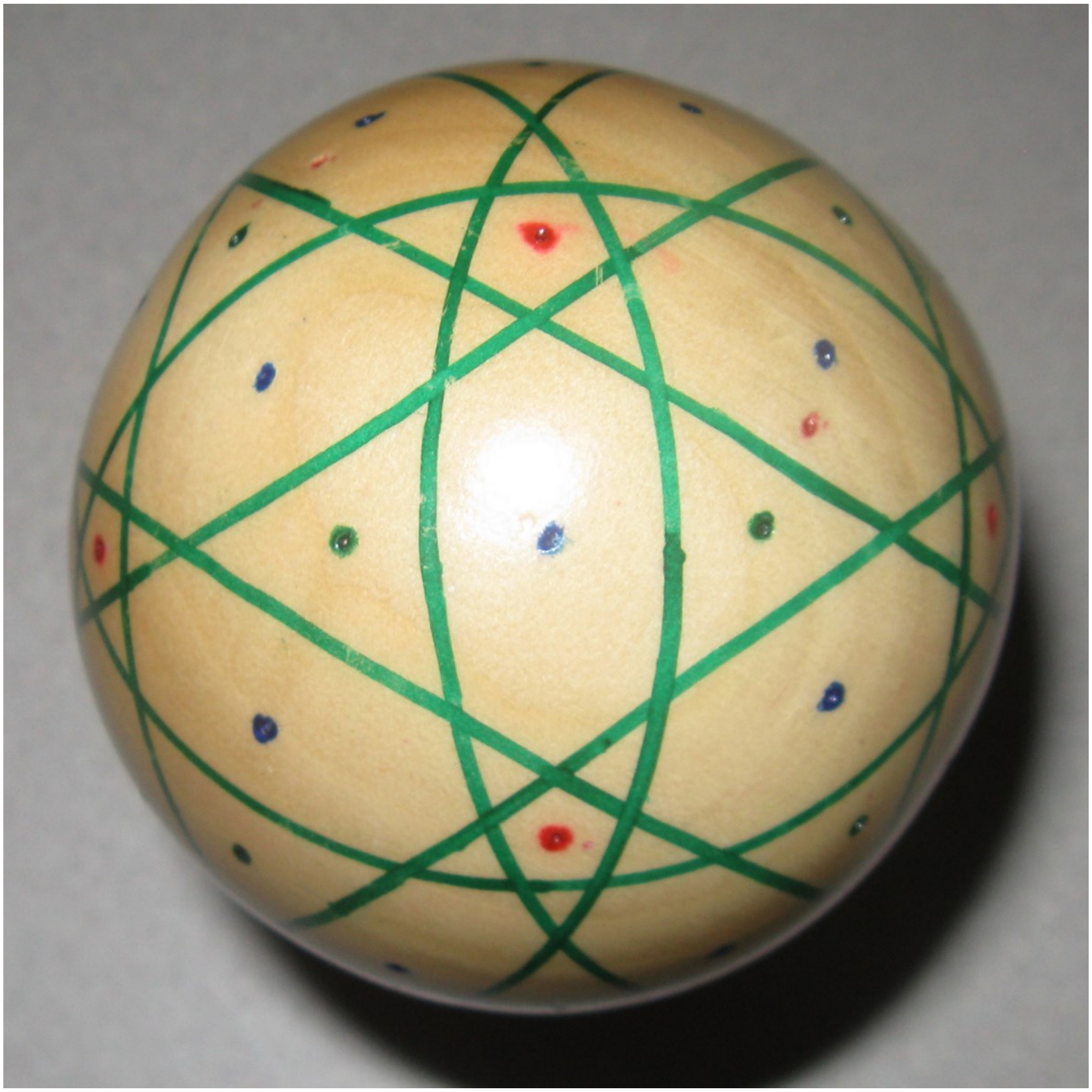


Figure 105

Figure 106

See Figure 107-109 for examples of a half step between **5V** and **$5V^{1e}$** viewed from the vertex, face and edge respectively. Notice that it also suggests a 62-sided solid (also known as a rhombicosidodecahedron).

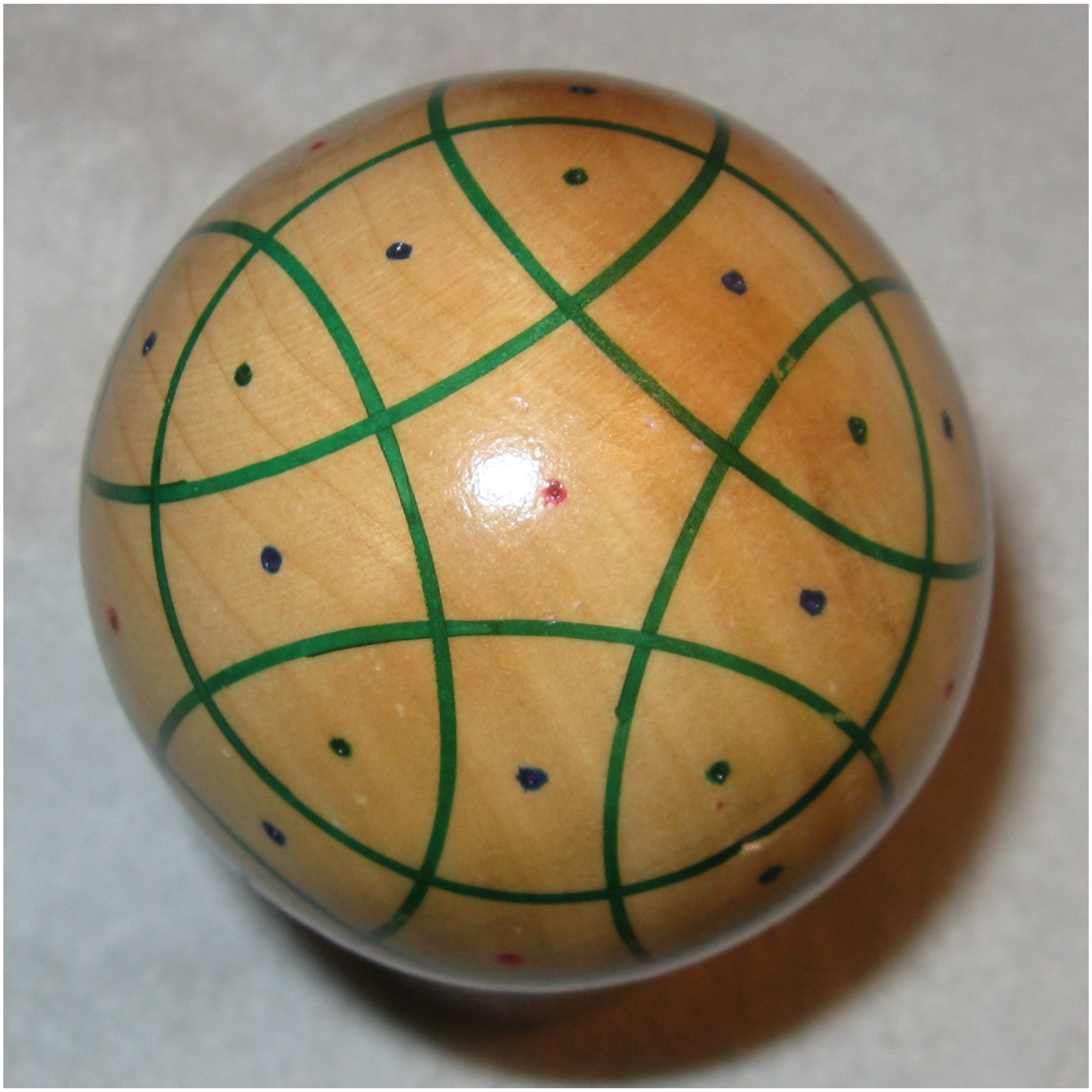


Figure 107

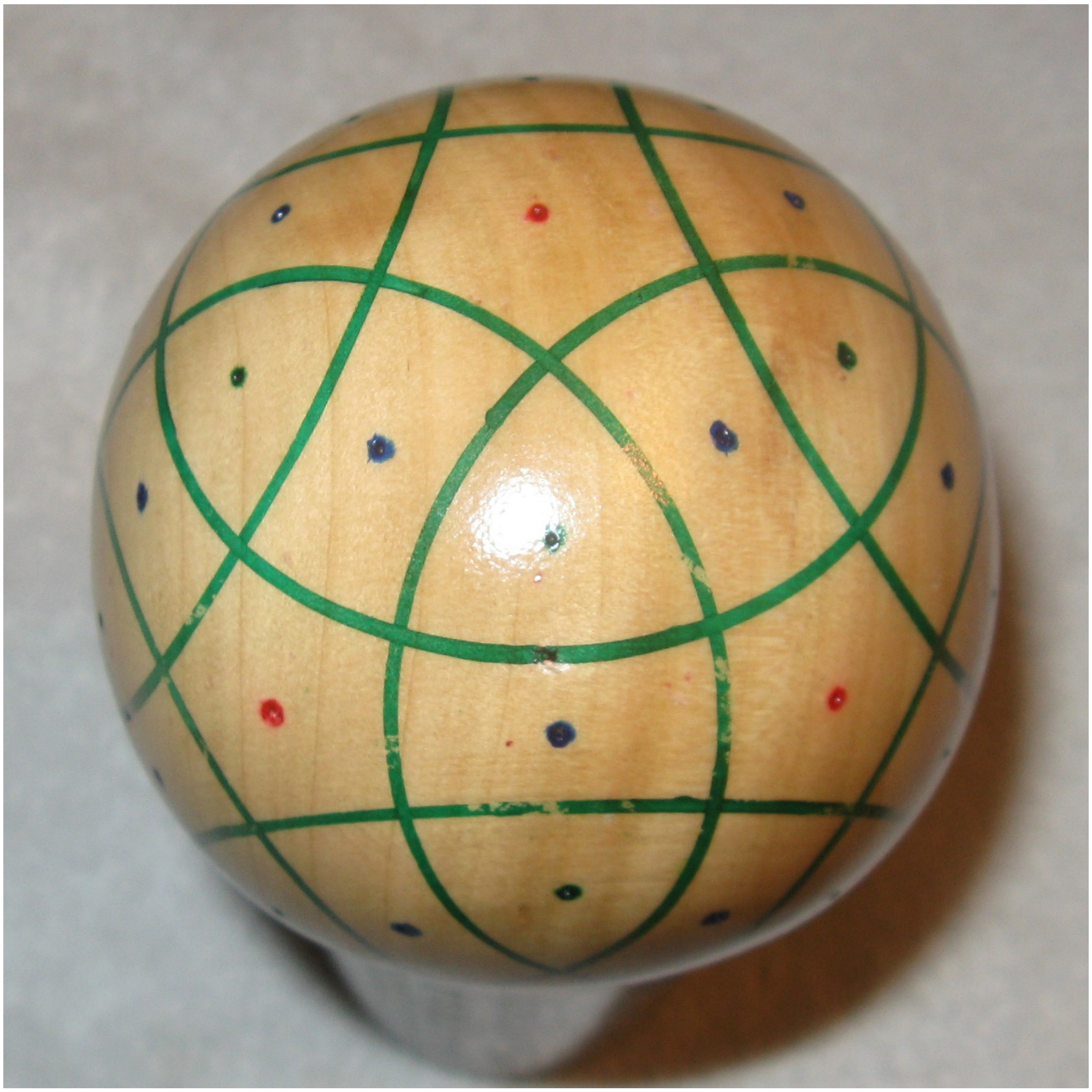


Figure 108

Figure 109

A harmonic arrangement can be created from an arrangement of several fractional steps. See an example of a 9-step arrangement between **5V** to **5V**^{3e} in Figure 110.

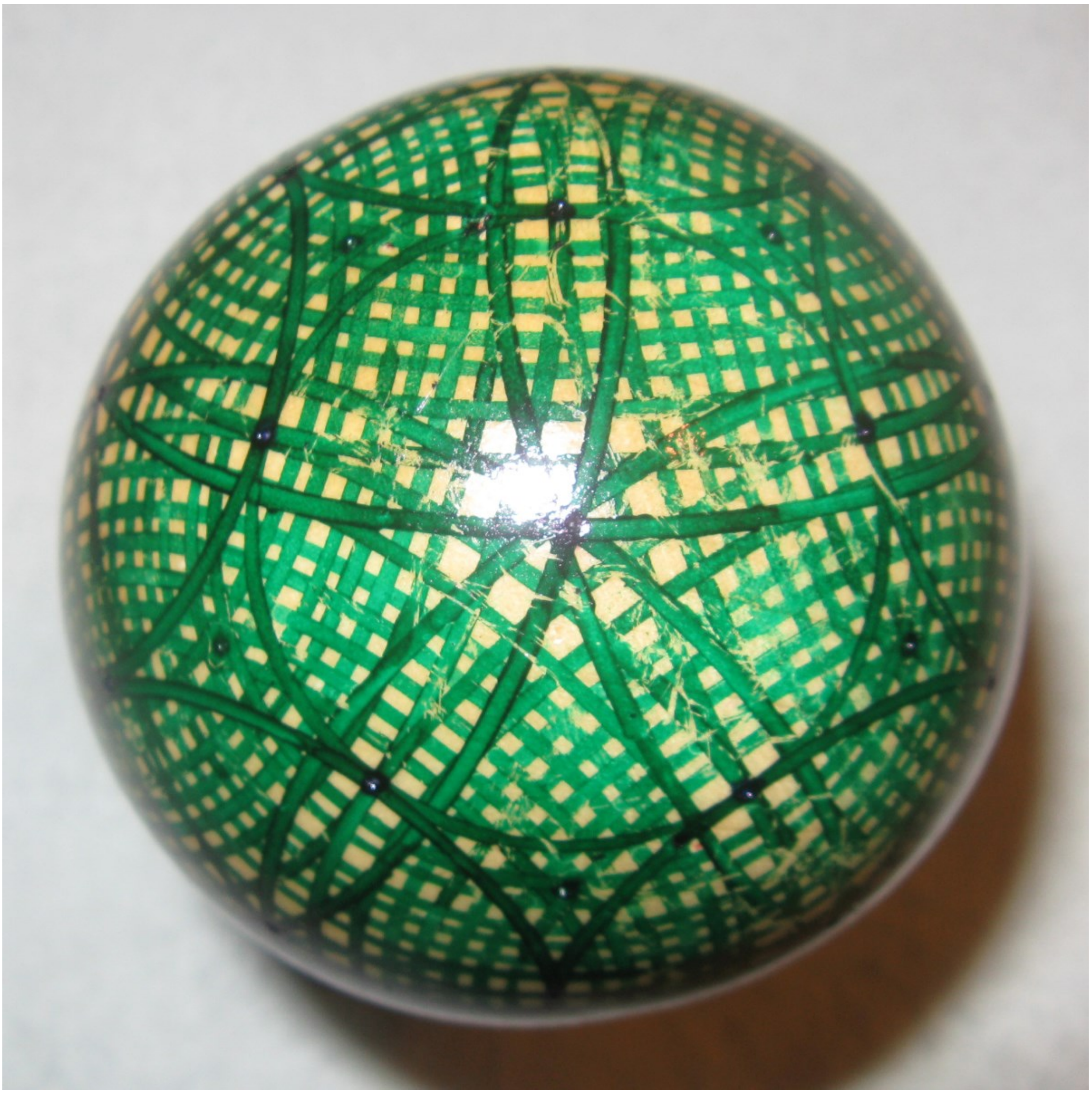


Figure 110

Appendix IV: Harmonic Inspired Art

Here are a few examples of some art works that is inspired by the 5th harmonic notes.

Appendix V: Polyconix Evolution

The arrangement of cones in each polyconix is an excellent example of form and function working together and creating something which is more than the sum of its parts. The adjacent cones in these polyconix are fully abutted in a stable arrangement that is secured in place by attractive forces. The cones have a form of geometric memory which allows the cones to almost self-assemble into the original polyconix.

What I also find remarkable is that groups of these free cones can bind together in many other ways than its original polyconix. I will explore some of these shapes using a hexaconix as an example. Six cones comprise a hexaconix and the cones are physically identical and as a result any six hexaconix cones will be able to create a hexaconix. There are a few small sub-component shapes consisting of a few hexaconix cones that have a relatively strong bond between cones. Refer to Figure 111-114 to see a random collection of single cones and sub-components.

I'm not implying any physical theory based on these shapes. This is just an example how this simple geometric model can evolve into more complex shapes from conix and cones.

All of the shapes illustrated in this Appendix were created with a construction set (Patent Pending 13/199,998) to model some random shapes.

Figure 111

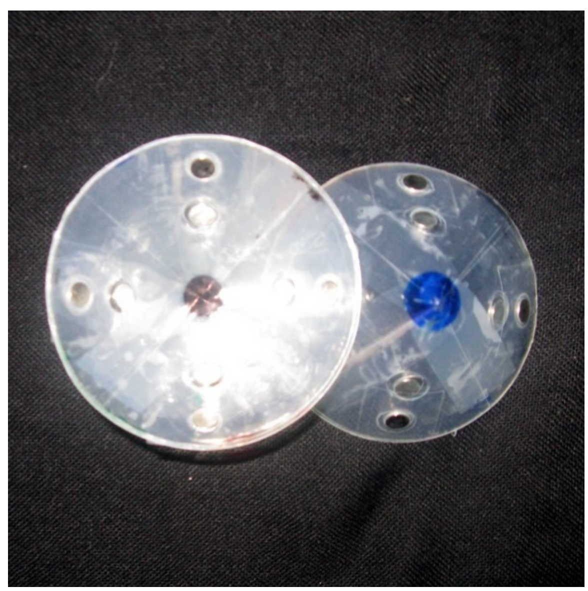


Figure 112

Figure 113

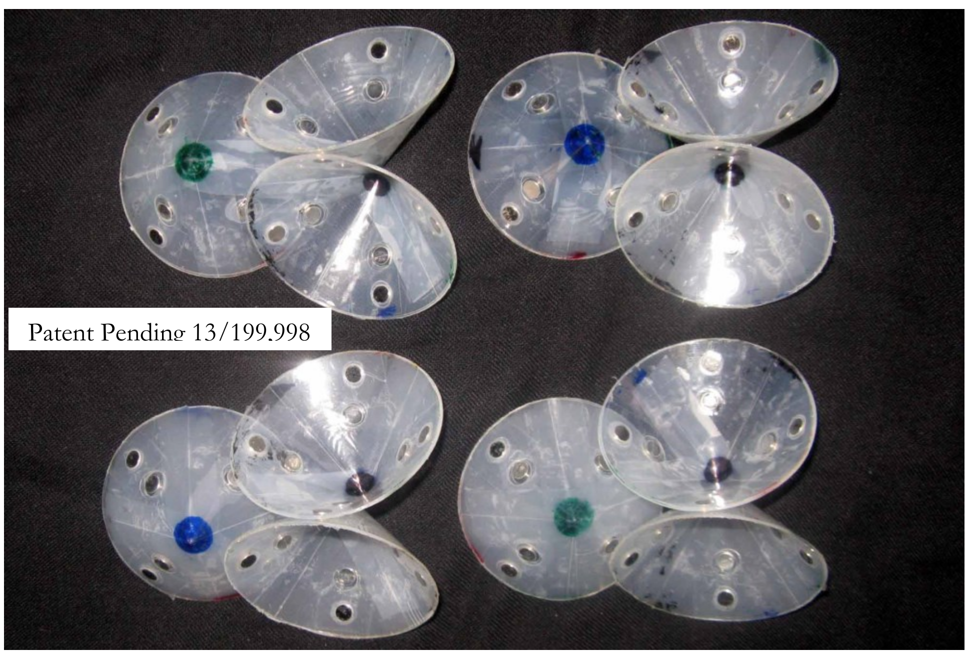


Figure 114

These sub-components make good building blocks for larger more complex shapes. In Figure 115-117, you will see a simple example of how a flat sheet can be created from several hexaconix cones. An unlimited number of hexaconix cones can from very large structures.

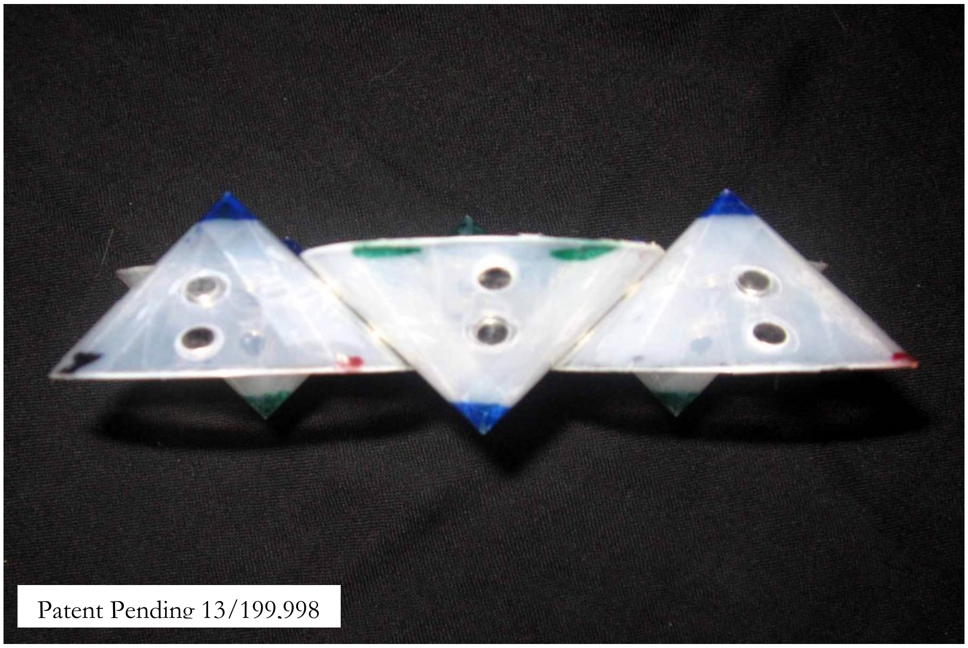


Figure 115

Figure 116

Figure 117

In these next examples, large rings and organic-like structures are illustrated in Figure 118-119.

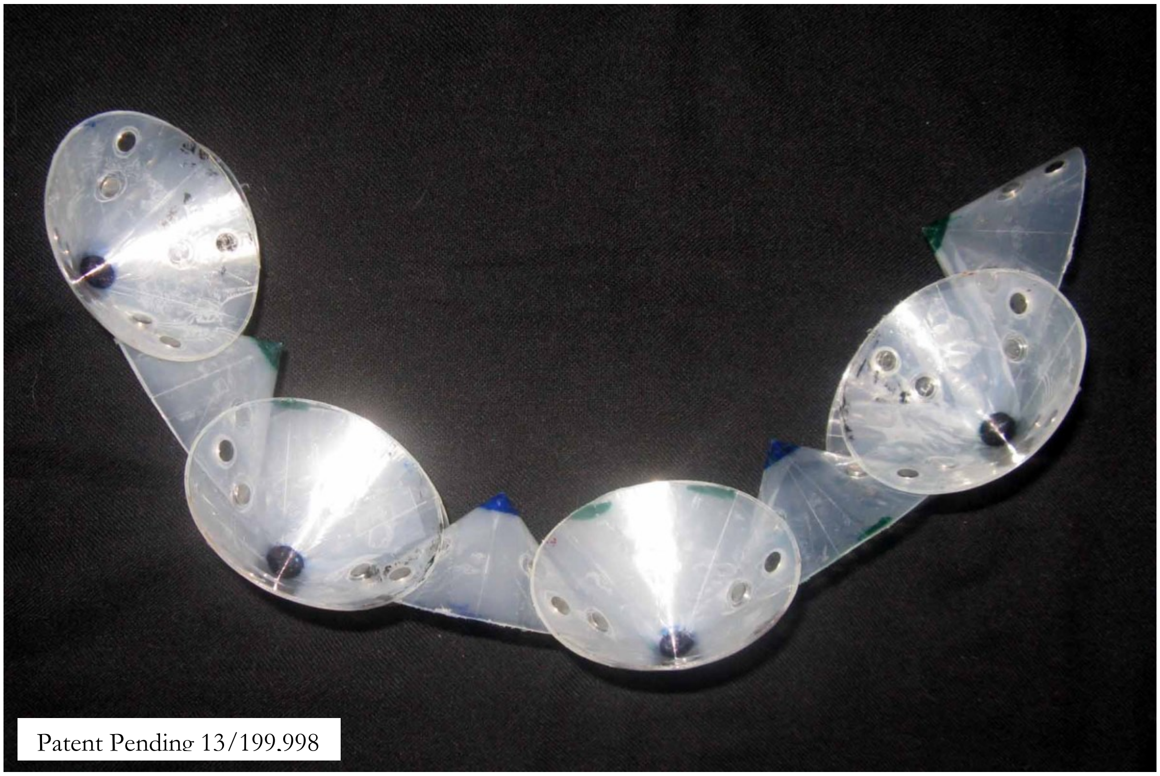


Figure 118

Figure 119

Hexaconix can be assembled with their cones' apexes pointing inwards or outwards refer to Figure 120. The inward pointing cones creates the basic hexaconix shape, which shares the geometric properties of a hexahedron. The outward pointing cones define a shape that shares the geometric properties of an octahedron, the dual of a hexahedron. Pyramid-like shapes can be created by the arrangement of hexaconix cones seen in Figure 121-122.

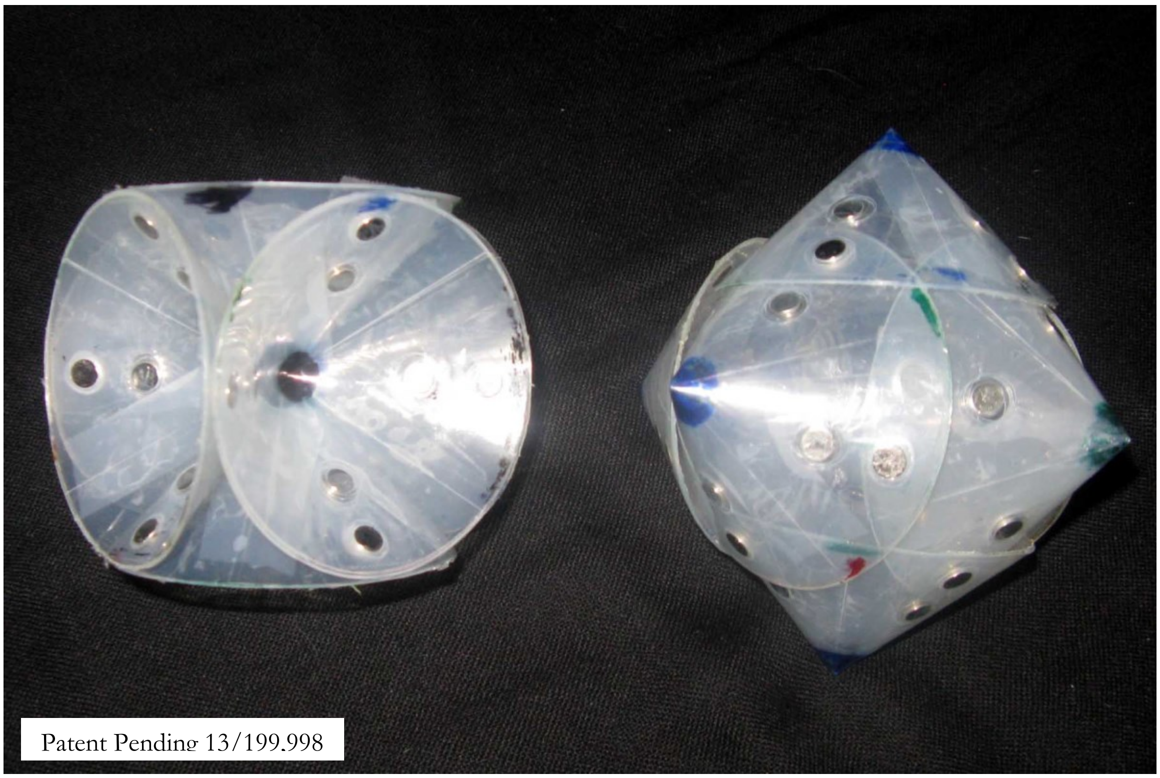


Figure 120

Figure 121

Figure 122

Hexaconix can also bond with other hexaconix in a few different ways, refer to Figure 123-125. In this way, long strings or tight lattice structures can result. The arrangement of hexaconix illustrated in Figure 123 would be a flexible structure, while the arrangement illustrated in Figure 124 would be a more rigid structure.

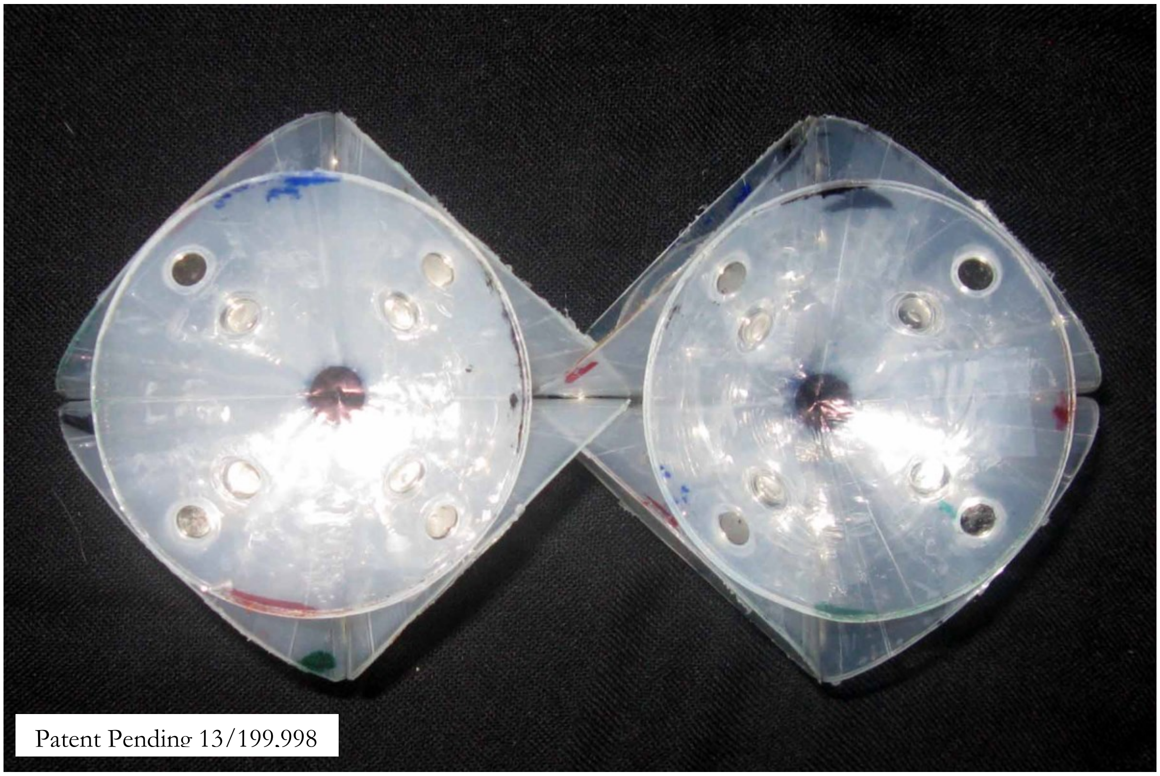


Figure 123

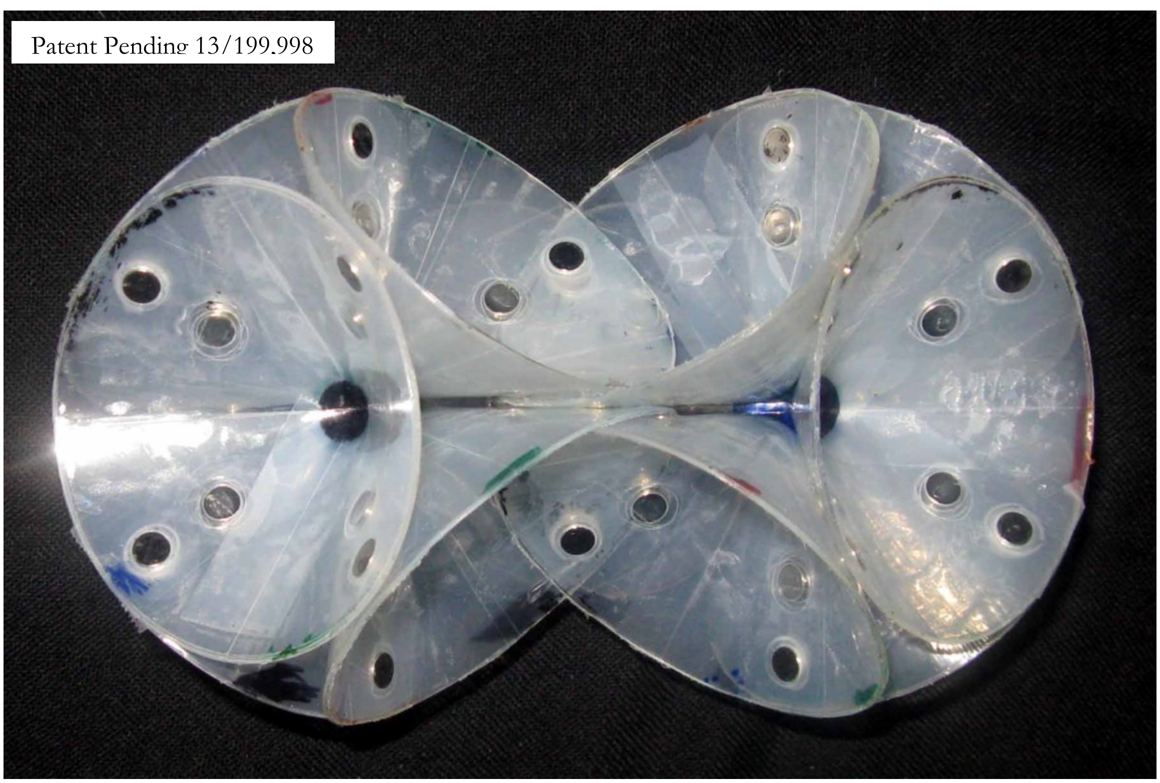


Figure 124

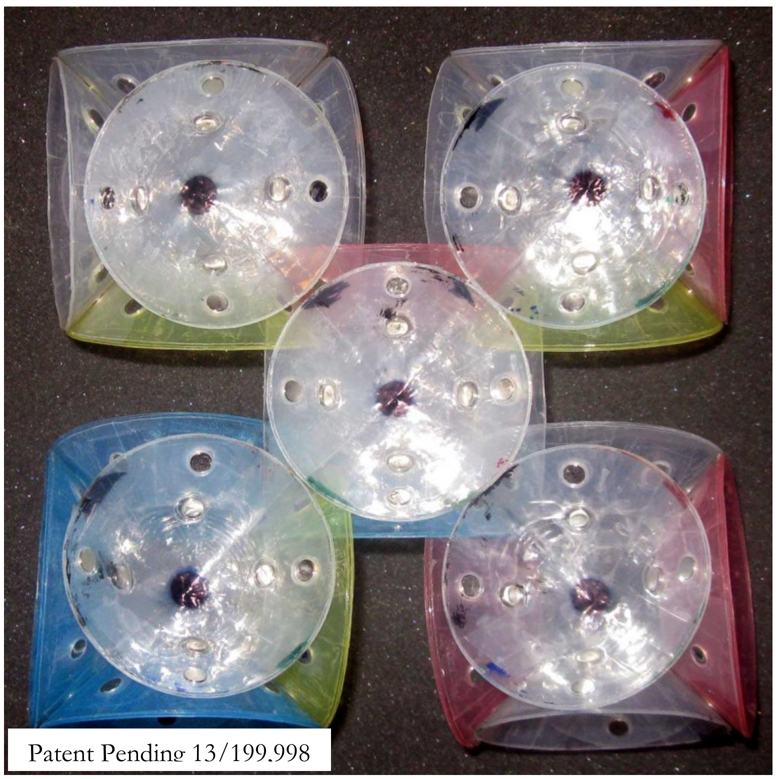


Figure 125

When the cones are truncated, hexaconix rings are created. These hexaconix rings are capable of creating even more complex shapes. Refer to Figure 126-128 to see the basic hexaconix shaped formed from hexaconix rings viewed from three different perspectives. Referring to the reference hexahedron, Figure 126 represents a view from its vertices axes, Figure127 represents a view from its edge axes and Figure 128 represents a view from its face axes. The universe now consists of a vast number of polyconix, polyconix cones, polyconix rings and other more complex but less stable shapes. See the following Figures 129-136 for a small sample of possible shapes that could result from the vast sea of hexaconix rings and cones.

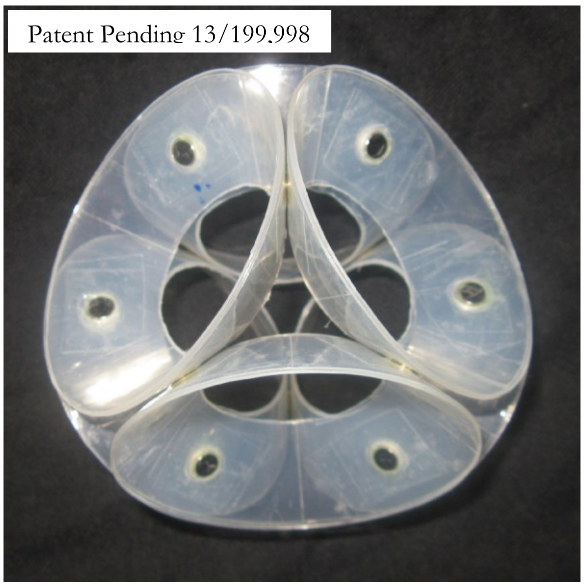


Figure 126

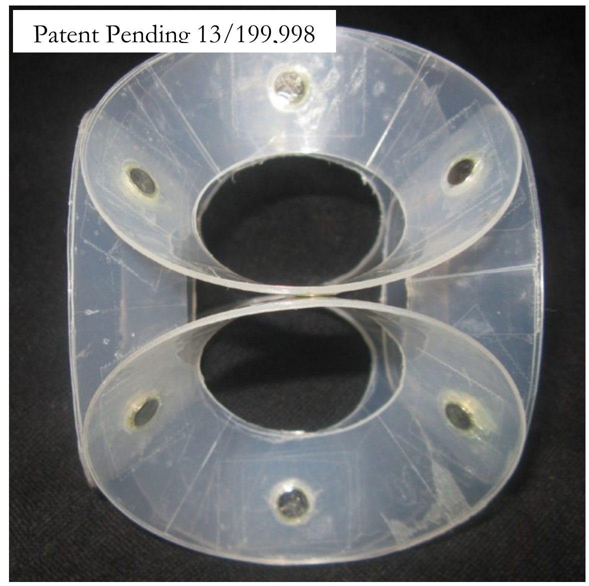


Figure 127

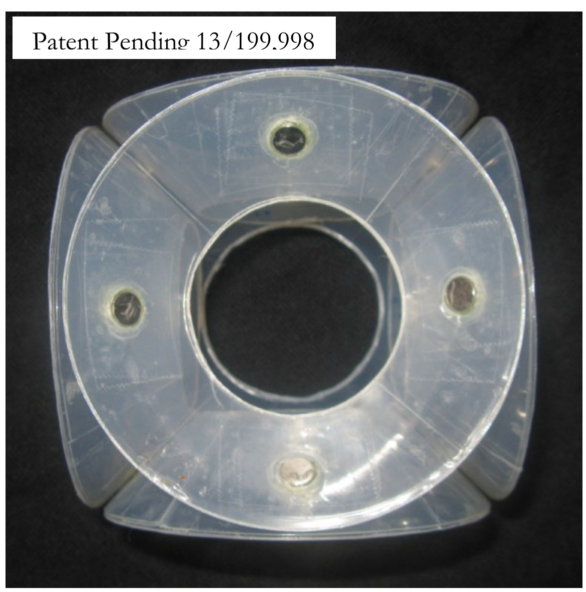


Figure 128

Figure 129

Figure 130

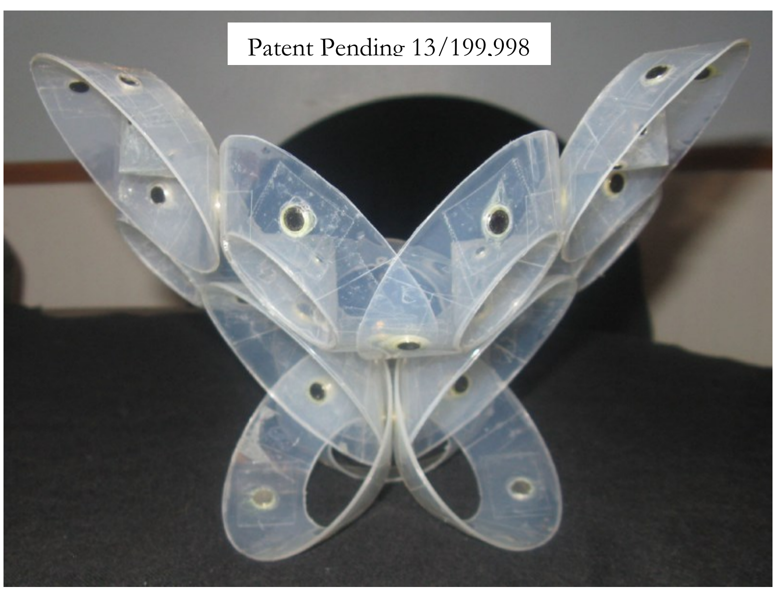


Figure 131

Figure 132

Figure 133

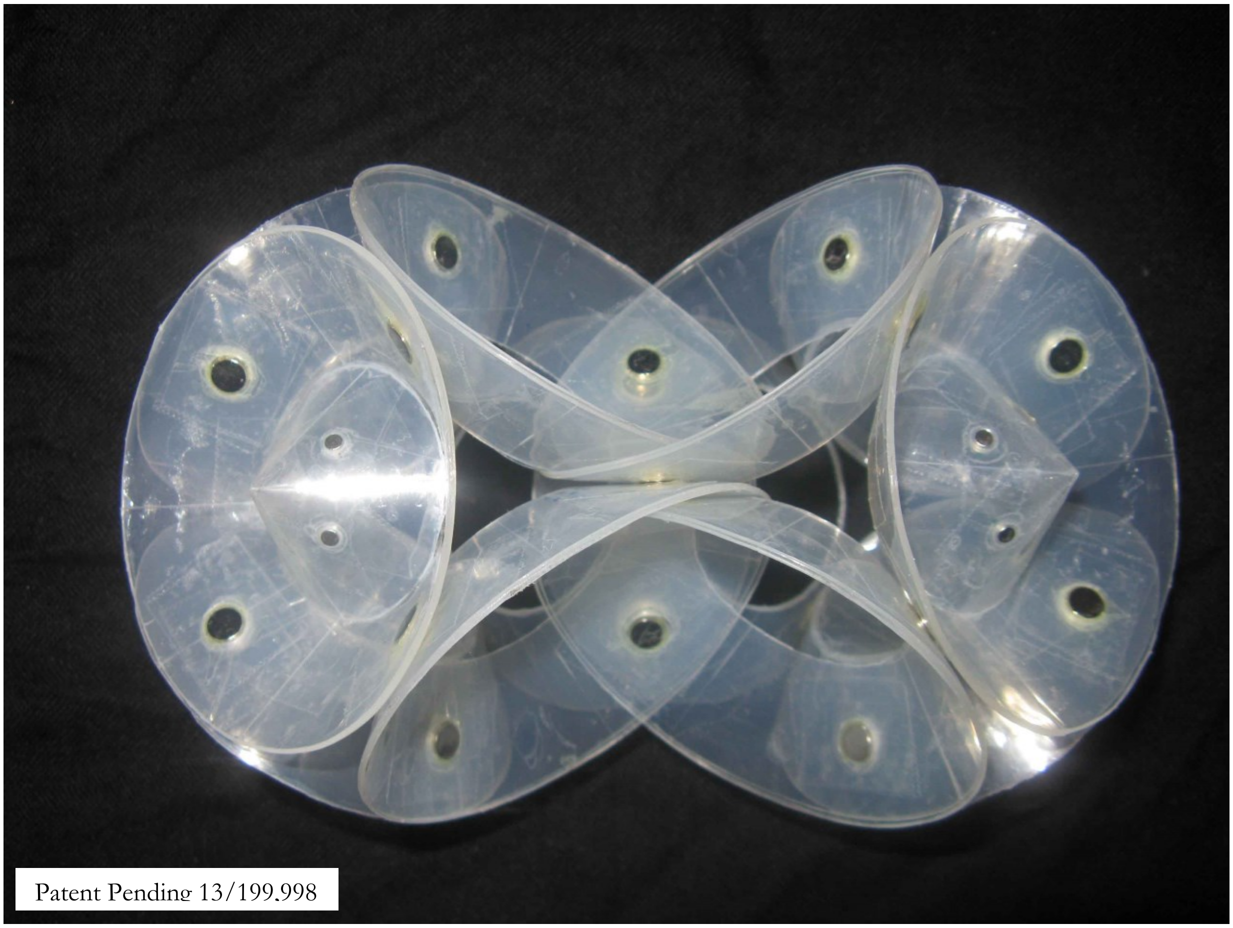


Figure 134

Figure 135

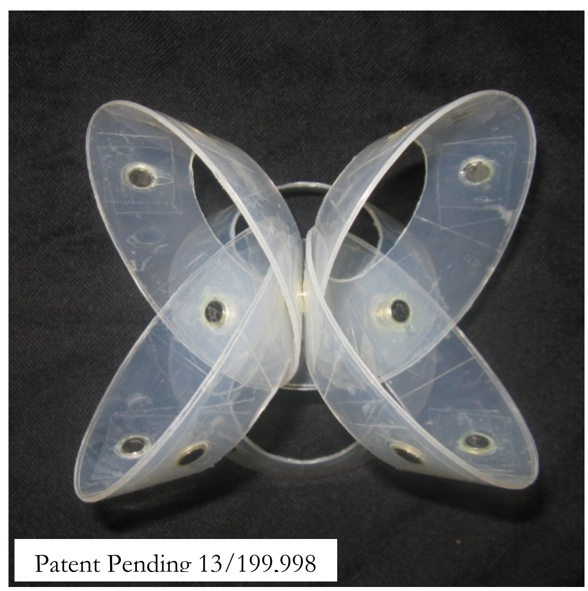


Figure 136

Figure 137

Figure 138

Figure 139

Figure 140

Figure 141

Figure 142

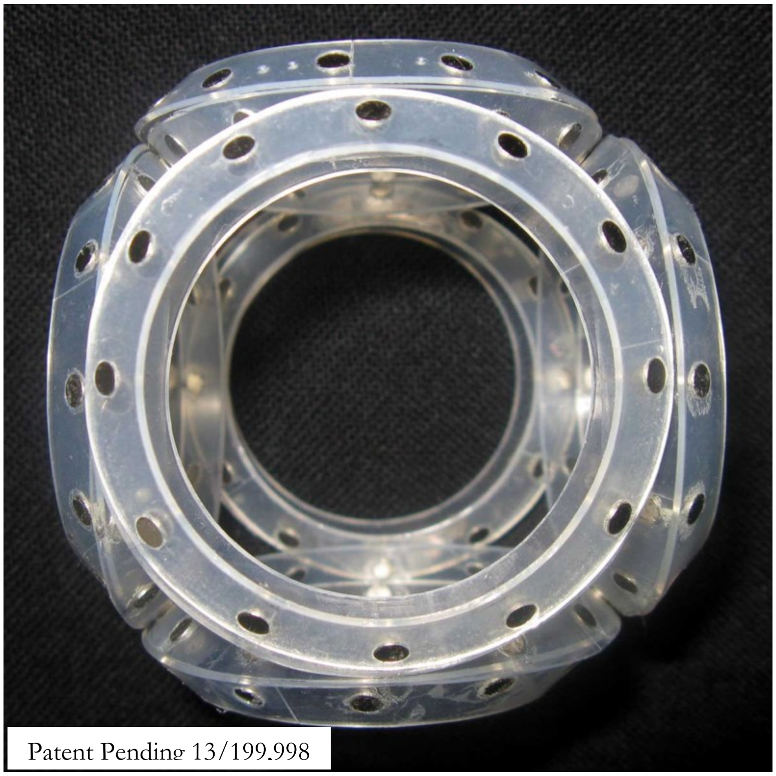


Figure 143

Figure 144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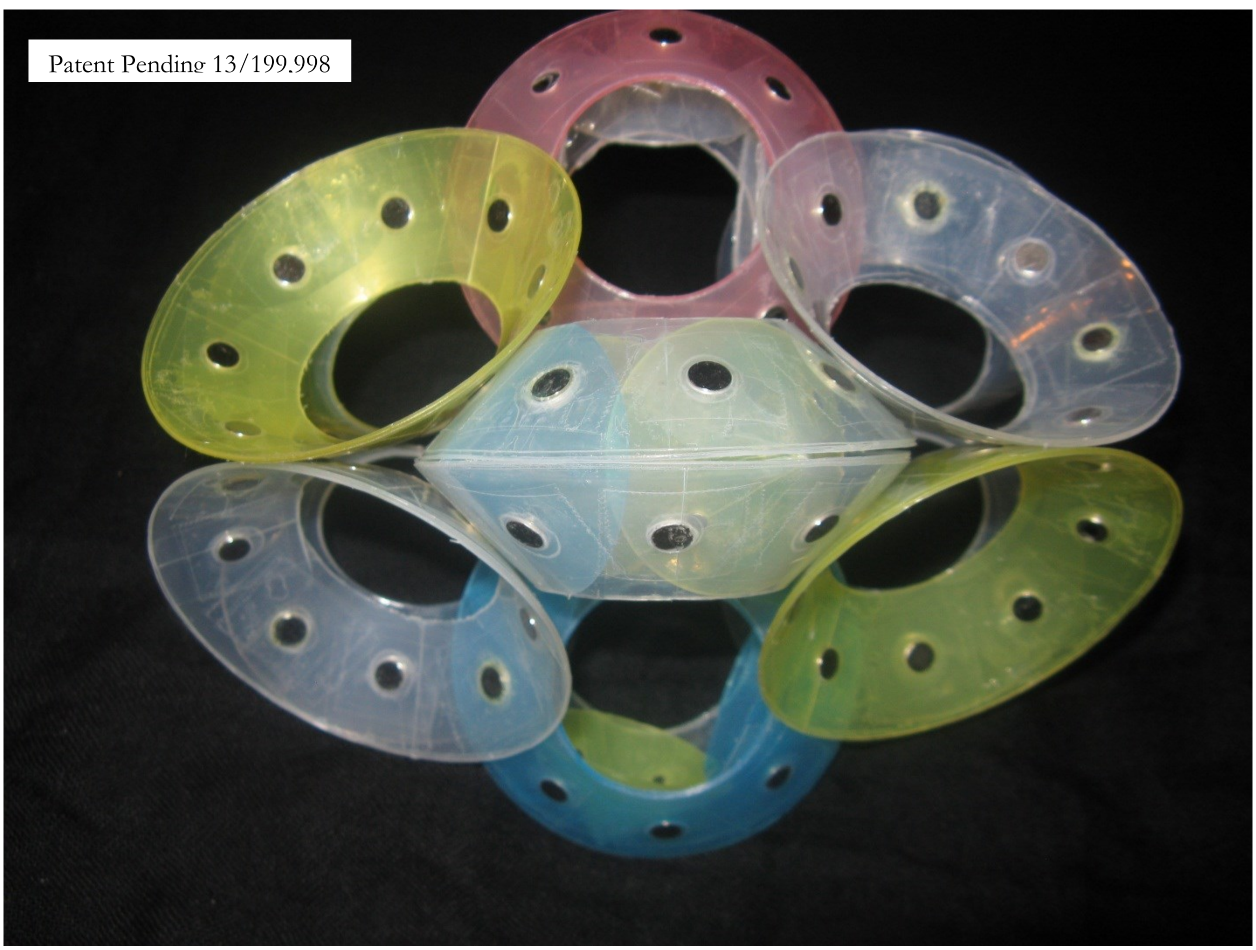
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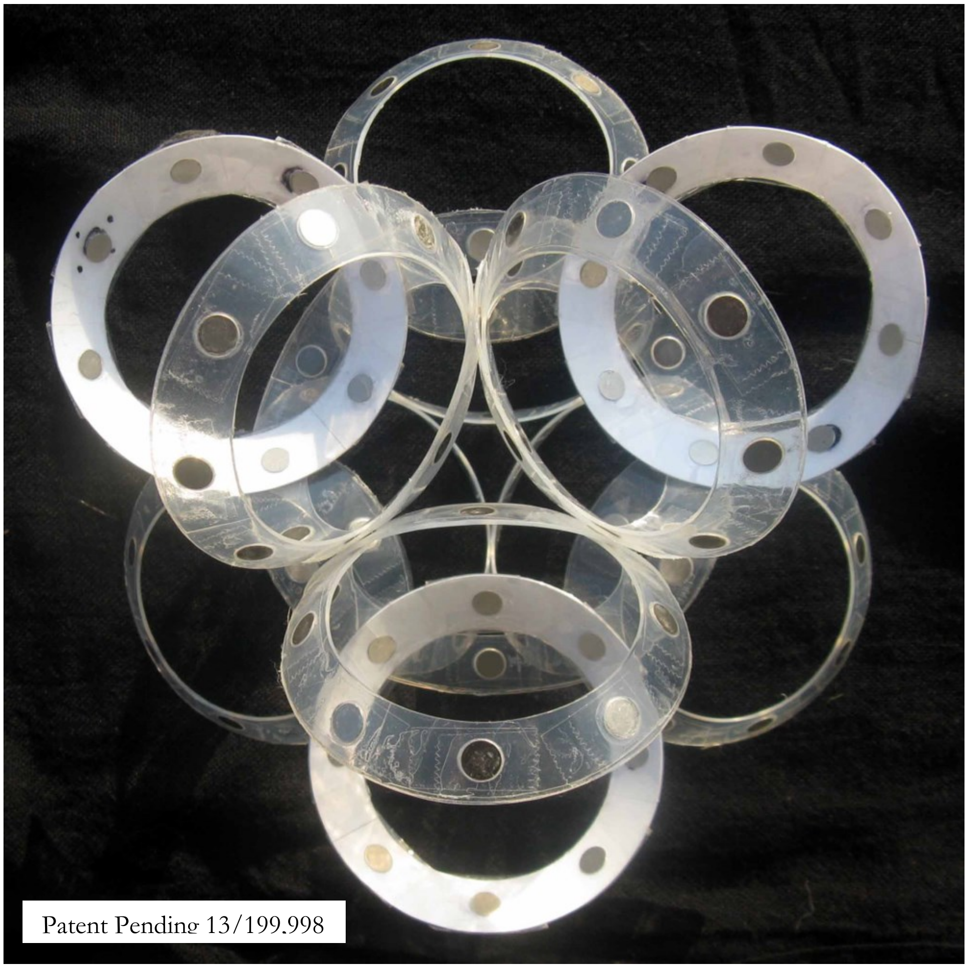
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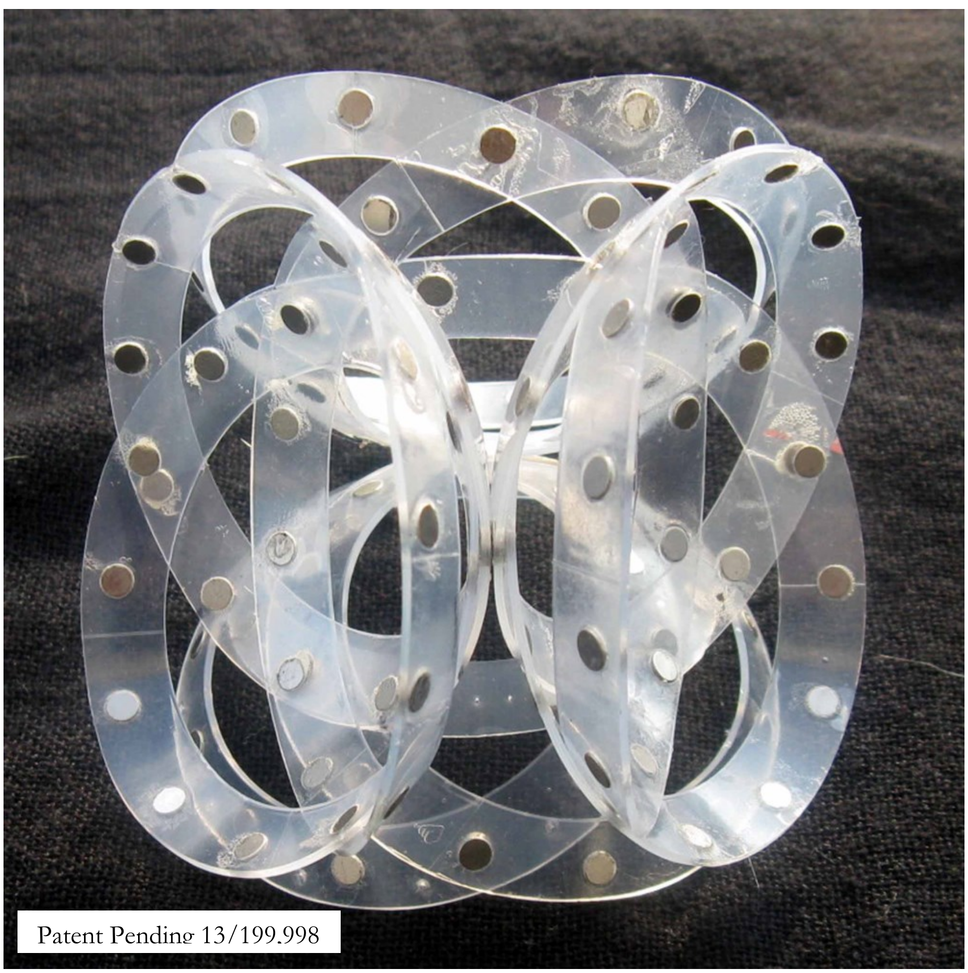
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